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Cartan connections and integrable vortex equations

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ABSTRACT

We demonstrate that integrable abelian vortex equations on constant curvature Riemann surfaces can be reinterpreted as flat non-abelian Cartan connections. By lifting to three dimensional group manifolds we find higher dimensional analogues of vortices. These vortex configurations are also encoded in a Cartan connection. We give examples of different types of vortex that can be interpreted this way, and compare and contrast this Cartan representation of a vortex with the symmetric instanton representation.

1. Introduction

In this paper we will finish the program started in [19], and furthered in [20], of relating all five of the integrable abelian vortex equations from [13] to the geometry of three dimensional Lie groups. In particular we demonstrate that an abelian vortex is equivalent to a flat non-abelian connection.

This relationship showcases the role that constant curvature geometries play in the construction of explicit vortex configurations. The most common way to see this relationship is to consider the Taubes equation satisfied by the modulus of the Higgs field, in the integrable cases it reduces to the Liouville equation. The Liouville equation is satisfied by the conformal factor for a metric with constant Gauss curvature. This leads to the interpretation of the modulus of the Higgs field as a conformal factor for a constant curvature metric. This metric is often called the Baptista metric [2].

The standard Abelian-Higgs model is a two dimensional model of gauged vortices. The model consists of a complex scalar field ϕ called the Higgs field and a $U(1)$ gauge potential a. On a Riemann surface M_0 with the conformal factor Ω_0, the Abelian-Higgs model at critical coupling has the static energy functional [15]

$$E = \frac{1}{2}\int \left(\frac{B^2}{\Omega_0^2} + \frac{1}{\Omega_0}\overline{D_i\phi}D^i\phi + \frac{1}{4}\left(1-|\phi|^2\right)^2 \right) d\mathrm{Vol}, \tag{1.1}$$

with $B = f_{12} = \partial_1 a_2 - \partial_2 a_1$. This can be rewritten using a Bogomol'nyi argument to see that the energy is bounded below,

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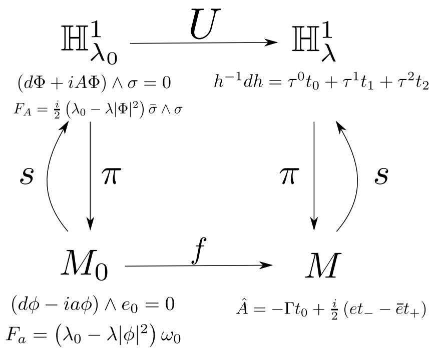

Fig. 1. This summarises the four sets of equations and spaces that we relate in this paper. An extension on the left hand side relating the equations on $\mathbb{H}^1_{\lambda_0}$ to vortex magnetic modes on flat spaces can be included when $\lambda_0 \neq 0$. This figure is adapted from one in [18].

$$E \geq \pi |N|, \tag{1.2}$$

with N the winding number of the field. When $N > 0$ the minimisers solve the first order, Bogomol'nyi, equations

$$D_{\bar{z}}\phi = (\partial_{\bar{z}} - ia_{\bar{z}})\,\phi = 0, \tag{1.3}$$

$$B = \frac{\Omega_0}{2}\left(1 - |\phi|^2\right), \tag{1.4}$$

called the vortex equations. For $N < 0$ the first equation, (1.3), becomes $D_z\phi = 0$.

Decomposing the Higgs field as $\phi = e^{u+i\chi}$, and taking account of the singularities of u at the zeros, Z_r $r = 1, \dots, N$ possibly repeated, of the Higgs field, the Bogomol'nyi equations can be converted into the Taubes equation

$$-\frac{4}{\Omega_0}\partial_z\partial_{\bar{z}}u = 1 - e^{2u} - \frac{4\pi}{\Omega_0}\sum_{r=1}^{N}\delta(z - Z_r). \tag{1.5}$$

A detailed study of this equation, for the case of the Abelian-Higgs model on the plane, is given in [9]. From a mathematical point of view this model is constructed from the data of a Riemann surface M_0, a connection a and a smooth complex section ϕ of a complex line bundle over M_0. The pair (ϕ, a) is called a vortex.

Fig. 1 summarises the general story followed in this paper, and demonstrates the relationship between vortex configurations on Lie groups in the top left and vortices on Riemann surfaces in the bottom left. It guides how we proceed in this paper starting with explaining the vortex story along the bottom line, before moving on to the three dimensional story which gives the details of the upper part of the figure.

The paper is ordered as follows. In Sec. 2 we state our conventions for the geometry of Lie groups and Riemann surfaces. Following this we demonstrate how to encode the structure and Gauss equations within a non-abelian flat connection, and how this flat connection descends from the Maurer-Cartan form on a Lie group. This describes the right hand side of Fig. 1.

Sec. 3 summarises results about the integrable abelian vortex equations of [13], as well as a discussion of the geometric interpretations of these vortices. These geometric interpretations are: deformations of the metric introducing degeneracies, an idea introduced in [2], and non-abelian Cartan connections encoding a degenerate co-frame.

Next in Sec. 4 we introduce and study the three dimensional generalisation of a vortex, a vortex configuration. Vortex configurations are also given by a flat connection, this time the pull back of the Maurer-Cartan one-form by a bundle map. This relationship between vortex configurations and flat connections is the key result of the section, and we use it to relate vortex configurations to vortices. Sec. 5 gives the construction of solutions to massless Dirac equations from vortices. It includes extensions of the results of [19,20] and suggestions for future work.

Then Sec. 6 compares the non-abelian connections introduced here to describe vortices with the symmetric instantons given in [3]. The explicit forms of both connections are given and evidence for a conjectured duality between the different vortex equations is discussed. Finally Sec. 7 summarises the paper and gives some future directions of research.

2. Lie groups and Cartan connections

2.1. Group conventions

To understand how to read Fig. 1 we first need to explain what the notation $\mathbb{H}^1_\lambda$ means. We are interested in the three Lie groups $SU(2)$, SE_2, and $SU(1,1)$, which are all groups of determinant one matrices. Respectively these are the group of determinant one 2×2 unitary matrices, the component of the Euclidean group in two dimensions connected to the identity, and the group of 2×2 pseudo-unitary matrices. For a more concrete realisation of the groups we take the generators of the Lie algebra $\mathfrak{h}^1_\lambda = \mathrm{Lie}\left(\mathbb{H}^1_\lambda\right)$ to be

$$t_0 = -\frac{i}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad t_1 = -\frac{i}{2}\begin{pmatrix}0 & -\lambda\\ 1 & 0\end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix}0 & \lambda\\ 1 & 0\end{pmatrix}, \tag{2.1}$$

with the commutation relations

$$[t_a, t_b] = C_{ab}^{\ \ c} t_c \qquad \text{with} \quad C_{01}^{\ \ 2} = 1, \quad C_{02}^{\ \ 1} = -1, \quad C_{12}^{\ \ 0} = -\lambda, \tag{2.2}$$

and all others vanishing. We use the notation $\mathbb{H}^1_\lambda$ for this group[1] with;

$$\mathbb{H}^1_{-1} = SU(2), \quad \mathbb{H}^1_0 = SE_2, \quad \mathbb{H}^1_1 = SU(1,1). \tag{2.3}$$

It is convenient to introduce the complex combinations

$$t_\pm = t_1 \pm i t_2 \tag{2.4}$$

which satisfy

$$[t_0, t_\pm] = \mp i t_\pm, \quad [t_+, t_-] = 2i\lambda t_0. \tag{2.5}$$

The first of these can be interpreted as t_0 defining a complex structure on its complement such that $(t_+)t_-$ is (anti)-holomorphic.

The group has inverse metric

$$g^{ab} = \mathrm{diag}(-\lambda, 1, 1) \tag{2.6}$$

which is used to raise and lower group indices, and is degenerate in the $\lambda = 0$ case.

As a submanifold of $\mathbb{C}^2$, $\mathbb{H}^1_\lambda$ is defined as

$$\mathbb{H}^1_\lambda = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 - \lambda |z_2|^2 = 1\}, \tag{2.7}$$

the signature of the submanifold depends on the sign of λ. The complex coordinates (z_1, z_2) parametrise a determinant one matrix $h \in \mathbb{H}^1_\lambda$ through

$$h = \begin{pmatrix} z_1 & \lambda \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \tag{2.8}$$

As $\mathbb{H}^1_\lambda$ is a Lie group there is a real, left-invariant Maurer-Cartan one-form encoding the geometry,

$$h^{-1}dh = \sigma^0 t_0 + \sigma^1 t_1 + \sigma^2 t_2. \tag{2.9}$$

We say they encode the geometry as they satisfy the structure equations

$$d\sigma^a = -\frac{1}{2} C_{bc}^{\ \ a} \sigma^b \wedge \sigma^c, \tag{2.10}$$

where the $C_{bc}^{\ \ a}$ are the structure constants of the Lie algebra. As a precursor to what we will consider later, observe that $h^{-1}dh$ can be viewed as a $\mathfrak{h}^1_\lambda$ valued connection on $\mathbb{H}^1_\lambda$, whose flatness is equivalent to the structure equations. Encoding the geometry of manifolds in terms of flat connections is a central theme of Cartan geometry, see the textbook [21] and the PhD thesis [23] for a general discussion of Cartan geometry.

Throughout it is convenient to work with the complex combinations

$$\sigma = \sigma^1 + i\sigma^2, \qquad \bar{\sigma} = \sigma^1 - i\sigma^2, \tag{2.11}$$

which obey

$$d\sigma = i\sigma \wedge \sigma^0, \qquad d\sigma^0 = \frac{i\lambda}{2}\sigma \wedge \bar{\sigma}. \tag{2.12}$$

In terms of the complex coordinates the left invariant one-forms have the explicit expressions

$$\sigma = 2i\,(z_1 dz_2 - z_2 dz_1)\,, \quad \sigma^0 = i\,(\bar{z}_1 dz_1 - \lambda \bar{z}_2 dz_2 - z_1 d\bar{z}_1 + \lambda z_2 d\bar{z}_2)\,. \tag{2.13}$$

In terms of the left invariant one-forms the metric and orientation[2] are

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[1] This group is equivalent to the group G_C considered in [3]. In [3] the authors pick the generators $J_a = -t_a^T$ and $C = -\lambda$. The conventions here are chosen so that for $\lambda = -1$ they match those in [19].

[2] The slightly unconventional ordering is so that it makes contact with the volume form on $\mathbb{R}^3$ after stereographic projection in the $\lambda = 1, -1$ cases.

$$ds^2 = \frac{1}{4}\left(-\frac{1}{\lambda}\left(\sigma^0\right)^2 + \left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right), \tag{2.14}$$

$$\mathrm{Vol}_{\mathbb{H}^1_\lambda} = \frac{1}{8}\sigma^1 \wedge \sigma^0 \wedge \sigma^2. \tag{2.15}$$

The metric is singular in the $\lambda = 0$ case but the only problem due to this is that we are unable to construct zero modes on $\mathbb{H}^1_0$ in Sec. 5.

The dual left-invariant vector fields, X_a, generate the right-action $h \to ht_a$ and have the commutators

$$[X_a, X_b] = C_{ab}^{\ \ c} X_c. \tag{2.16}$$

In terms of the combinations

$$X_\pm = X_1 \pm iX_2 \tag{2.17}$$

we have

$$[X_0, X_\pm] = \mp i X_\pm, \qquad [X_+, X_-] = 2\lambda i X_0. \tag{2.18}$$

In terms of the complex coordinates the left invariant vector fields take the form

$$X_0 = -\frac{i}{2}\left(z_1\partial_1 + z_2\partial_2 - \bar{z}_1\bar{\partial}_1 - \bar{z}_2\bar{\partial}_2\right), \tag{2.19}$$

$$X_- = -i\left(\bar{z}_1\partial_2 + \lambda\bar{z}_2\partial_1\right), \tag{2.20}$$

$$X_+ = \overline{X_-}, \tag{2.21}$$

where we have used $\partial_i = \frac{\partial}{\partial z_i}$. The only non-zero pairing are

$$\sigma^0(X_0) = 1, \qquad \sigma(X_-) = \bar{\sigma}(X_+) = 2. \tag{2.22}$$

A key feature of $\mathbb{H}^1_\lambda$ is that it is a circle fibration over a Riemann surface M with constant Gauss curvature $K = -\lambda$. For $\lambda = -1$ this is the familiar Hopf fibration, while in the other cases we have a trivial bundle. The projection is

$$\pi : \mathbb{H}^1_\lambda \to M, \qquad h \mapsto z = \frac{z_2}{z_1}, \tag{2.23}$$

with z a local complex coordinate on M. There is also the familiar section, local when $\lambda = -1$ but global otherwise,

$$s : z \mapsto \frac{1}{\sqrt{1-\lambda|z|}}\begin{pmatrix} 1 & \lambda\bar{z} \\ z & 1 \end{pmatrix}. \tag{2.24}$$

This enables us to relate our Maurer-Cartan one-form to the Cartan connection for the Riemann surface M. Here we use the same definition of a Cartan connection as in [24], in fact since our Riemann surfaces are all homogeneous spaces we are actually dealing with Kleinian geometries.

Definition 2.1. A Klein geometry (G, H) is a pair of a Lie group G and a closed subgroup $H \subset G$ such that the quotient G/H is smooth and connected.

From [21] this is equivalent to having a principal H-bundle $G \to G/H$, which is equipped with a natural flat connection, the Maurer-Cartan one-form on G. A Cartan geometry is essentially a manifold that is locally, or infinitesimally, Kleinian. Klein geometries are the simplest examples of Cartan geometries, where the Cartan connection is flat and the manifold is globally a homogeneous space.

The group manifold $\mathbb{H}^1_\lambda$ is not simply connected when $\lambda = 0, 1$. This is because topologically $\mathbb{H}^1_{-1} = S^3$, $\mathbb{H}^1_0 = \mathbb{R}^2 \times S^1$, and $\mathbb{H}^1_1 = H^2 \times S^1$. The generator of the fundamental group is the curve

$$\gamma = \{e^{\varphi t_0} \in \mathbb{H}^1_\lambda | \varphi \in [0, 4\pi)\}. \tag{2.25}$$

This curve is contractible when $\lambda = -1$ since $\pi_1(\mathbb{H}^1_\lambda) = \pi_1(S^3) = 0$.

When it is non-contractible, only flat connections with a prescribed holonomy around γ can be globally trivialised. We will encounter this constraint when discussing vortices on the group manifold in Sec. 4.

2.2. Two dimensional geometry

On a Riemann surface M with constant Gauss curvature K we work in local complex coordinates z. As we are considering M to be either S^2, H^2 or $\mathbb{R}^2$, z is a global coordinate except on S^2. The Riemann surface has metric[3]

$$ds^2 = \Omega dz d\bar{z} = \frac{4}{\left(1+K|z|^2\right)^2} dz d\bar{z}. \tag{2.26}$$

This metric admits the (local) complexified co-frame

$$e = \frac{2dz}{1+K|z|^2}. \tag{2.27}$$

The geometry of the Riemann surface is encoded in the structure and Gauss equations,

$$de - ie \wedge \Gamma = 0 \tag{2.28}$$

$$d\Gamma = \mathcal{R} = \frac{i}{2} K e \wedge \bar{e} \tag{2.29}$$

where

$$\Gamma = iK \frac{z d\bar{z} - \bar{z} dz}{1+K|z|^2} \tag{2.30}$$

is the spin connection one-form and $\mathcal{R}$ is the curvature two-form. When we have two Riemann surfaces we denote the one with Gauss curvature K by M and the one with Gauss curvature K_0 by M_0. There are corresponding 0 subscripts on the co-frame fields, spin connection, and curvature two-form.

There are two results worth noting here. The first is that the structure and Gauss equations can be interpreted as the flatness of a Cartan connection $\hat{A}$. The second relates $\hat{A}$ to the Maurer-Cartan one-form on $\mathbb{H}^1_\lambda$ when $\lambda = -K$.

Proposition 2.2. *The structure and Gauss equations,* (2.28) *and* (2.29)*, for the co-frame,* (2.27)*, and spin connection* (2.30)*, are equivalent to the flatness of the* $\mathfrak{h}^1_\lambda$ *valued connection*

$$\hat{A} = -\Gamma t_0 + \frac{i}{2}\left(e t_- - \bar{e} t_+\right), \tag{2.31}$$

where $K = -\lambda$ *is the Gauss curvature.*

Note that $\hat{A}$ is a connection on the principal $U(1)$ bundle

$$\mathbb{H}^1_\lambda \to M = \mathbb{H}^1_\lambda / U(1). \tag{2.32}$$

The proof is a straightforward computation of the curvature of $\hat{A}$.

Proof. The curvature of $\hat{A}$ is

$$F_{\hat{A}} = d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}], \tag{2.33}$$

$$= -\left(\mathcal{R} + \lambda \frac{i}{2} e \wedge \bar{e}\right) t_0 + \frac{i}{2}\left(de - i\Gamma \wedge e\right) t_- - \frac{i}{2}\left(d\bar{e} + i\Gamma \wedge \bar{e}\right) t_+. \tag{2.34}$$

The vanishing of the coefficient of t_0 is equivalent to the Gauss equation, (2.29) with curvature $K = -\lambda$, and the vanishing of the $t_\pm$ coefficients is equivalent to the structure equations, (2.28). □

Proposition 2.3. *Using the local section* (2.24) *the Cartan connection* $\hat{A}$ *on* M, (2.31)*, is trivialised as the pullback of the Maurer-Cartan one-form,* (2.9)*,*

$$\hat{A} = s^*\left(h^{-1} dh\right). \tag{2.35}$$

Proof. To prove this use the explicit expression for the section, (2.24), to compute $s^*\sigma = ie$ and $s^*\sigma^0 = -\Gamma$. Then directly computing the pullback of Eq. (2.9) leads to $\hat{A}$. □

[3] Note that with this choice of metric $\mathbb{R}^2$ with $K=0$ has $ds^2 = 4dzd\bar{z}$. This may seem an unusual choice but it facilitates the same language to be used for all three surfaces.

Going the other way round $h^{-1}dh$ can be expressed in terms of $\pi^*\hat{A}$, where π is the bundle projection of Eq. (2.23). Unfortunately, this is only true up to a singular gauge transformation as

$$\pi^* e = -i\frac{\bar{z}_1}{z_1}\sigma, \qquad \pi^*\Gamma = -\sigma^0 + id\ln\left(\frac{z_1}{\bar{z}_1}\right). \tag{2.36}$$

These propositions convey a key concept of this work; we encode equations on a Riemann surface as flat connections, and relate these to the Maurer-Cartan one-form on $\mathbb{H}^1_\lambda$.

3. Integrable vortex equations

3.1. Vortices on Riemann surfaces

We turn now to vortex equations on Riemann surfaces. The vortex solutions of Eqs. (1.3) and (1.4) are known as hyperbolic vortices, since the equations are integrable on H^2. These hyperbolic vortex equations were generalised in [13] to give integrable vortex equations on a more general Riemann surface M_0. The hyperbolic case has a long history with the first solutions given in [25], while the spherical case $M_0 = S^2$ was first considered in [12,17], but the language was unified in [13] to give five[4] integrable vortex equations. The equations can be written down on any Riemann surface but the integrability relies on having constant curvature.

The vortex equations involve two parameters, suggestively called λ_0 and λ, and describe a pair (ϕ, a) of a connection a and a section ϕ of a complex line bundle over M_0. When M_0 is non-compact, appropriate asymptotics, $|\phi| \to 1$ on ∂M_0, need to be imposed to ensure finite energy [9,15].

Definition 3.1. A (λ_0, λ) vortex is a pair (ϕ, a) of a connection, a, and a section, ϕ, of a complex line bundle over M_0 which satisfy the (λ_0, λ) vortex equations

$$(d\phi - ia\phi)\wedge e_0 = 0 \qquad F_a = da = \left(\lambda_0 - \lambda|\phi|^2\right)\omega_0. \tag{3.1}$$

Here

$$\omega_0 = \frac{i}{2}e_0 \wedge \bar{e}_0, \tag{3.2}$$

is the Kähler form on M_0.

From [13] solutions are given by rational maps $f : M_0 \to M$ where M_0, M are Riemann surfaces with constant Gauss curvature $-\lambda_0, -\lambda$ respectively. There is the following direct way to solve (3.1). Define the Higgs field and connection as

$$f^*e = \phi e_0, \qquad a = f^*\Gamma - \Gamma_0. \tag{3.3}$$

Then pulling back Eq. (2.28) by f gives the first vortex equation, while pulling back the Gauss equation (2.29), noting $f^*(e\wedge\bar{e}) = |\phi|^2 e_0 \wedge \bar{e}_0$ and using the Gauss equation on M_0, leads to the second vortex equation. The data of a holomorphic map between two constant curvature Riemann surfaces is thus all we need to construct a solution to the vortex equations.

Another way to show this is to reduce the vortex equations to the Liouville equation. Decomposing the Higgs field as $\phi = e^{u+i\chi}$ leads to a generalisation of the Taubes equation

$$-\frac{4}{\Omega_0}\partial_z\partial_{\bar{z}}h = \left(\lambda_0 - \lambda e^{2h}\right) - \frac{2\pi}{\Omega_0}\sum_{r=1}^{N}\delta\left(z - Z_r\right), \tag{3.4}$$

with Ω_0 the conformal factor on M_0 and $Z_r \in \mathbb{C}$ the zeros of ϕ. A scaling argument from [13] shows that there are five integrable cases:

- Hyperbolic vortices $\lambda_0 = \lambda = 1$,
- Popov vortices $\lambda_0 = \lambda = -1$,
- Jackiw-Pi vortices $\lambda_0 = 0$, $\lambda = -1$,
- Ambjørn-Olesen vortices $\lambda_0 = 1 = -\lambda$,
- Bradlow vortices $\lambda_0 = 1$, $\lambda = 0$.

As noted above, the case of $\lambda_0 = \lambda = 0$, sometimes called Laplace vortices, could also be included. This naturally fits into the framework that we discuss here. Explicit expressions for ϕ and a in terms of f are given in [13].

[4] Or should that be 6 equations? The $\lambda = \lambda_0 = 0$ case was not considered in [13] but was called the Laplace vortex equation in [3]. It corresponds to a covariantly holomorphic section ϕ and a flat connection a. These Laplace vortices fit into the Cartan framework given here.

3.2. Baptista metric

The idea of interpreting vortices geometrically stems from [14] where the Higgs field of a hyperbolic vortex is represented as the ratio of conformal factors

$$|\phi|^2 = \frac{f^*\Omega}{\Omega_0}\left|\frac{df}{dz}\right|^2 . \tag{3.5}$$

Then in [2] it was shown that a vortex defines a degenerate conical geometry on M_0, where the metric has conformal factor $|\phi|^2\Omega_0$ relative to the flat metric $dzd\bar{z}$. This conformal factor is zero at the vortex centres. This idea was extended in [13] where it is referred to as the Baptista metric. In the integrable cases the Baptista metric is given by

$$ds_B^2 = f^*ds^2 = |\phi|^2 ds_0^2. \tag{3.6}$$

This says that a vortex defines a degenerate co-frame on M_0, with the vortex equations forming part of the structure and Gauss equations for this co-frame.

It is important to be aware that since the Baptista metric has the conformal factor $|\phi|^2$ it is degenerate at the N, not necessarily distinct, zeros of the Higgs field. As observed in [2] the Riemann curvature two-form associated with the metric is extended to the zeros by adding delta function singularities

$$\mathcal{R}' = \mathcal{R}_0 + F_a - 2\pi\sum_{j=1}^{N}\delta_{Z_j}, \tag{3.7}$$

where we use δ_{Z_j} for the two-form Dirac delta supported on the point Z_j.

This can be understood as the Baptista metric having a conical singularity with surplus angle $2\pi N_j$ at a zero of multiplicity N_j, with $N=\sum_j N_j$. The local geometry around the point Z_j thus resembles a ruffled collar and is sometimes called an Elizabethan geometry. For the case of Popov vortices on S^2, a is a connection on a line bundle of even degree, $N=2n-2$ with $n=1,2,\ldots$ etc, and thus

$$\int_{S^2} F_a = 4\pi n - 4\pi . \tag{3.8}$$

This is cancelled by the integral over the delta functions and thus

$$\int_{S^2}\mathcal{R}' = \int_{S^2}\mathcal{R}_{S^2} = 4\pi , \tag{3.9}$$

which can be interpreted as the Gauss-Bonnet theorem holding for $\mathcal{R}'$. This is in contrast to the pullback of the curvature two-form, $f^*\mathcal{R}_{S^2}$, which integrates to $4\pi n$ since in this case the map $f: S^2 \to S^2$ has degree n [13,19].

For the other types of vortex we still have Equation (3.7), and F_a still integrates to $2\pi N$, once the appropriate boundary conditions are taken into account, which is again cancelled by the delta function contribution. However, as H^2 and $\mathbb{R}^2$ are non-compact the integrals of the curvature forms are not defined. One way to get around this is to work on compact manifolds covered by H^2 or $\mathbb{R}^2$, such as a Riemann surface of genus $g>1$ which is the quotient of H^2 by a Fuchsian group $\Lambda < SU(1,1)$. This complicates the story somewhat so we do not focus on it here.

This example shows that the spin connection of the degenerate co-frame ϕe_0, $\tilde{\Gamma}$ differs from the pulled back spin connection $f^*\Gamma$ by a contribution due to the zeros of ϕ and this contribution is what leads to the singularities in $\mathcal{R}'$.

3.3. Vortices as flat connections

From the discussion of the Baptista metric and Proposition 2.2 it follows that the vortex equations are equivalent to the flatness of the non-abelian connection $f^*\hat{A}$. The key observation is that pulling back the co-frame field from M, $f^*e=\phi e_0$, defines a degenerate co-frame from the data of the vortex, and the structure and Gauss equations for this co-frame imply the vortex equations of Eq. (3.1).

Corollary 3.2. *Given the flat connection $\hat{A}$ defined in Eq. (2.31), and a holomorphic map $f: M_0 \to M$, the flatness of*

$$f^*\hat{A} = -\left(a+\Gamma_{\lambda_0}\right)t_0 + \frac{i}{2}\left(\phi e_{\lambda_0}t_- - \bar{\phi}\bar{e}_{\lambda_0}t_+\right), \tag{3.10}$$

is equivalent to the (λ_0,λ) vortex equations.

As $\hat{A}$ is a connection on a $U(1)$ bundle over M, its pullback is a connection on the pulled back bundle over M_0 Since f is a rational function it can be written as $f(z) = f_2(z)/f_1(z)$ for two holomorphic functions f_1, f_2. The Higgs field ϕ, and thus the connection $f^*\hat{A}$ have zeros, and potentially singularities, determined by the f_i. To see this explicitly note that

$$f^*e = \phi e_0 = 2\frac{f_2' f_1 - f_1' f_2}{|f_1|^2 + K|f_2|^2}\frac{\bar{f}_1}{f_1}dz, \tag{3.11}$$

which has zeros at the ramification points of f, and singularities at the zeros of f_1. There are singularities when the co-frame e has a singularity, this happens when $M = S^2$ as the co-frame is local in a patch with coordinate singularity at one of the poles of the sphere. Under f the pre-images of the coordinate singularity of e become the zeros of f_1.

For example if q is a zero of f_1 then near q

$$\phi e_0 \sim A\frac{\bar{z} - \bar{q}}{z - q}dz, \tag{3.12}$$

for a constant A. When ϕe has singularities they are inherited by the vortex Cartan connection $f^*\hat{A}$. Thus it is not really defined on the total space $\mathbb{H}^1_\lambda$ but on

$$P = \mathbb{H}^1_\lambda \backslash \bigcup_j \pi^{-1}(q_j), \tag{3.13}$$

where the q_j are the zeros of f_1. For the case of Popov vortices an extensive discussion of the singularities and their properties is given in [19].

In the language of Cartan connections our results so far are that $\hat{A}$ is a gauge potential for the Cartan connection describing the geometry of M, while $f^*\hat{A}$ is a gauge potential for a Cartan connection describing the deformed geometry on M_0 due to the vortex (ϕ, a).

In all six cases the data of a vortex is encoded in a rational map. However, the specific details of the rational map is related to the geometry of the Riemann surfaces. When $M = S^2$ f has poles and when $M_0 = \mathbb{R}^2, H^2$ the boundary condition $|\phi| \to 1$ is applied. Thus for Popov vortices $f : S^2 \to S^2$ is a ratio of polynomials and has both zeros and poles, for Jackiw-Pi vortices $f : \mathbb{R}^2 \to S^2$ is a based rational map satisfying $\lim_{z\to\infty} f(z) \to 0$ which means that the degree of the denominator is larger than the degree of the numerator [7]. While for hyperbolic vortices $f : H^2 \to H^2$ is a bounded holomorphic function on hyperbolic space, and thus is represented by a finite Blashke product with no poles in the unit disc [25]. Bradlow vortices correspond to $f : H^2 \to \mathbb{R}^2$ and similar to the hyperbolic case can be expressed as Blashke products. The Ambjørn-Olesen case is similar to the hyperbolic case except that $f : H^2 \to S^2$ is now a finite Blashke product that can have poles, the argument for why this is true is given in Sec. 4.2.2. Finally, Laplace vortices have $f : \mathbb{R}^2 \to \mathbb{R}^2$, the only bounded examples are constant functions so the boundary condition would need to be relaxed to give interesting solutions.

4. Vortices on Lie groups

4.1. Vortex configurations

The next important actors in this story are vortex configurations on $\mathbb{H}^1_{\lambda_0}$. These are the generalisation of vortices to three dimensional group manifolds. Unlike vortices they do not involve sections and connections but give a way of writing the vortex equations in terms of one-forms and complex functions on the total space of the line bundle that vortices were defined on.

Before discussing vortex configurations we need to understand the equivariant functions on $\mathbb{H}^1_\lambda$.

Definition 4.1. The space of equivariant functions over $\mathbb{H}^1_\lambda$ is defined to be

$$C^\infty\left(\mathbb{H}^1_\lambda, \mathbb{C}\right)_N = \{F : \mathbb{H}^1_\lambda \to \mathbb{C} \mid 2iX_0F = NF\}, \quad N \in \mathbb{Z}. \tag{4.1}$$

In [19] the discussion of equivariant functions followed that in [10,11] with $N \in \mathbb{N}^0$. This is because for $SU(2)$, equivariant functions are functions on the Lens space $S^3/\mathbb{Z}_N$ and are related to sections of the hyperplane bundle. In general it is not a priori clear that we need to impose the same restriction that the degree is a non-negative integer. In practice we only encounter equivariant functions constructed from holomorphic polynomials in z_1, z_2 and for these $N \in \mathbb{N}^0$.

A short computation using the local section s, (2.24), results in the following commutative diagram

$$\begin{array}{ccc} C^\infty(\mathbb{H}^1_\lambda, \mathbb{C})_N & \xrightarrow{X_+} & C^\infty(\mathbb{H}^1_\lambda, \mathbb{C})_{N+2} \\ \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle s^*} \\ C^\infty(M_\lambda) & \xrightarrow[i(q\bar{\partial} - \lambda\frac{N}{2}z)]{} & C^\infty(M_\lambda) \end{array} \tag{4.2}$$

where $q=(1-\lambda|z|^2)$. The fact that for $\lambda=0$ the vector field X_+ is the lift of $\bar{\partial}$ is not particularly surprising as in this case $q=1$, $X_+=iz_1\bar{\partial}_2$ and the section s identifies z_2 with z.

As we will be dealing with two, potentially different, group manifolds $\mathbb{H}^1_{\lambda_0}$ and $\mathbb{H}^1_{\lambda}$ we require separate notation for the geometric objects on the two spaces. We use the notation s_a, σ^a, X_a for the generators, left-invariant one-forms and left-invariant vector fields respectively of the source group three manifolds, $\mathbb{H}^1_{\lambda_0}$, and t_a, τ^a, Y_a for the same objects on the target group three manifolds, $\mathbb{H}^1_{\lambda}$. Due to the details of our construction the Cartan connection is always valued in the Lie algebra of the target group.

Definition 4.2. Let A be a one-form on $\mathbb{H}^1_{\lambda_0}$ and $\Phi:\mathbb{H}^1_{\lambda_0}\to\mathbb{C}$ a complex function. We call the pair (Φ, A) a vortex configuration if the vortex equations,

$$(d\Phi+iA\Phi)\wedge\sigma=0 \qquad F_A=\frac{i}{2}\left(\lambda_0-\lambda|\Phi|^2\right)\bar{\sigma}\wedge\sigma, \tag{4.3}$$

with $F_A=dA$, are satisfied.

In the $\lambda=\lambda_0=-1$ case of [19] normalisation and equivariance conditions were included in the definition of vortex configurations. However, in the other cases $A(X_0)$ is not necessarily an integer so we do not include a normalisation condition. When we construct examples of vortex configurations they will be normalised. Equivariance conditions, when A is normalised, follow from the vortex equations, (4.3) and Cartan's identity. They are

$$\mathcal{L}_{X_0}\Phi=-iA(X_0)\Phi, \tag{4.4}$$

$$\mathcal{L}_{X_0}A=dA(X_0). \tag{4.5}$$

These vortex equations possess the $U(1)$ gauge invariance

$$(\Phi, A)\mapsto(e^{-i\beta}\Phi, A+d\beta), \qquad \beta\in C^\infty(\mathbb{H}^1_{\lambda_0}). \tag{4.6}$$

The three dimensional vortex equations in (4.3) have clear similarities to the vortex equations in (3.1). The left-invariant one-forms are the analogue of the complexified co-frame and spin connection. The precise relationship was given in the proof of Proposition 2.3 and in the discussion afterwords.

We now come to the central theorem of the paper, a method for constructing vortex configurations from bundle maps. Here a bundle map is a fibre-preserving morphism of the bundles covering a map between the bases.

Theorem 4.3. *A vortex configuration on $\mathbb{H}^1_{\lambda_0}$ determines a gauge potential for a flat $\mathfrak{h}^1_{\lambda}$ connection of the form*

$$\mathcal{A}=\left(A+\sigma^0\right)t_0+\frac{1}{2}\Phi\sigma t_-+\frac{1}{2}\bar{\Phi}\bar{\sigma}t_+. \tag{4.7}$$

Conversely, a flat $\mathfrak{h}^1_{\lambda}$ connection $\mathcal{A}$ on $\mathbb{H}^1_{\lambda_0}$ such that

$$\mathcal{A}(X_0)=pt_0, \qquad \mathcal{A}(X_-)=\alpha t_0+\Phi t_-, \tag{4.8}$$

for functions $p:\mathbb{H}^1_{\lambda_0}\to\mathbb{R}$ and $\alpha, \Phi:\mathbb{H}^1_{\lambda_0}\to\mathbb{C}$ determines a vortex configuration (Φ, A) through the expansion (4.7).

Given $\mathcal{A}$, a gauge potential for a flat Lie $\left(\mathbb{H}^1_{\lambda}\right)$ connection on $\mathbb{H}^1_{\lambda_0}$ of the form (4.7) which satisfies

$$\int_\gamma \mathcal{A}=2\pi n t_0, \tag{4.9}$$

for $n\in\mathbb{Z}$ and γ the curve defined in Eq. (2.25), it can be trivialised as $U^{-1}dU$ for a bundle map $U:\mathbb{H}^1_{\lambda_0}\to\mathbb{H}^1_{\lambda}$ which covers a holomorphic map $f:M_0\to M$. Without loss of generality U can be taken to have the form

$$U:(z^1,z^2)\mapsto\frac{1}{\sqrt{|F_1|^2-\lambda|F_2|^2}}\begin{pmatrix}F_1 & \lambda\bar{F}_2\\ F_2 & \bar{F}_1\end{pmatrix} \tag{4.10}$$

with $F_i:\mathbb{H}^1_{\lambda_0}\to\mathbb{C}$ where $|F_1|^2>\lambda|F_2|^2$.

The vortex configuration can be extracted from the bundle map as

$$\Phi\sigma=U^*\tau, \qquad A=U^*\tau^0-\sigma^0. \tag{4.11}$$

This result generalises Theorem 3.2 from [19] and Theorem 3.2 from [20], the result when $\lambda_0 = \lambda = -1$ or 1 respectively. Note that when $\lambda \neq -1$ we can assume that $F_1 \neq 0$ since $|F_1|^2 > \lambda|F_2|^2$. However when $\lambda = -1$ this assumption is not valid. This relates to the fact that when $\lambda = -1$ the map $f = s^*\left(\frac{F_2}{F_1}\right)$, which U covers, can have poles since it is a map $f : M_0 \to S^2$.

Proof. Given a $\mathfrak{h}^1_\lambda$ connection on $\mathbb{H}^1_{\lambda_0}$ in the vortex gauge, (4.7), the flatness condition $d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$ is equivalent to

$$\left(d\left(\Phi\sigma\right) + i\left(A + \sigma^0\right)\Phi\right) = 0 \qquad dA = \frac{i}{2}\left(\lambda_0 - \lambda|\Phi|^2\right)\bar{\sigma} \wedge \sigma. \tag{4.12}$$

Using equation (2.12) these are seen to be equivalent to the vortex equations, (4.3).

For the converse expand the flat $\mathfrak{h}^1_\lambda$ connection, $\mathcal{A}$, on $\mathbb{H}^1_{\lambda_0}$ in terms of the generators, t_0, t_+, t_-. The coefficients are linear combinations of σ^0, σ and $\bar{\sigma}$, as they form a basis of the co-tangent space of $\mathbb{H}^1_{\lambda_0}$. Imposing the conditions in Equation (4.8) leads to the gauge potential $\mathcal{A}$ being in the vortex form, Equation (4.7), with Higgs field Φ, and abelian gauge potential

$$A = (p-1)\sigma^0 + \frac{1}{2}\left(\alpha\sigma + \bar{\alpha}\bar{\sigma}\right). \tag{4.13}$$

The same calculation as above then gives that the vortex equations, (4.3), are satisfied.

To globally trivialise a flat connection $\mathcal{A}$ on $\mathbb{H}^1_{\lambda_0}$ in terms of $U : \mathbb{H}^1_{\lambda_0} \to \mathbb{H}^1_\lambda$ as $\mathcal{A} = U^{-1}dU$, its path-ordered exponential must be path independent. If this is the case U can be constructed explicitly from $\mathcal{P}\exp\left(\int_{\tilde{\gamma}}\mathcal{A}\right)$ along any path $\tilde{\gamma}$, starting at a fixed but arbitrary base point, [1].

As $\mathcal{A}$ is a flat connection the non-abelian Stokes theorem implies that the path-ordered exponential is path independent for contractible paths. The conditions in (4.8) ensure that the path-ordered exponential of $\mathcal{A}$ along γ coincides with the exponential of the ordinary integral. Then (4.9) implies that

$$\mathcal{P}\exp\left(\int_\gamma \mathcal{A}\right) = \mathbb{I}. \tag{4.14}$$

Flatness of the connection $\mathcal{A}$ and the non-abelian Stokes theorem then combine to give that the path-ordered exponential of $\mathcal{A}$ along any closed curve in $\mathbb{H}^1_{\lambda_0}$ is the identity. This gives the path independence of the path-ordered exponential.

The final part of the proof is to show that for $\mathcal{A} = U^{-1}dU$ satisfying (4.8) U is a bundle map covering a holomorphic function $M_0 \to M$. The first condition in (4.8) becomes

$$X_0 U = pUt_0, \tag{4.15}$$

with $p : \mathbb{H}^1_{\lambda_0} \to \mathbb{R}$. This is just the infinitesimal statement that U maps the fibres of $\mathbb{H}^1_{\lambda_0} \to M_0$ to the fibres of $\mathbb{H}^1_\lambda \to M$, in other words that U is a bundle map.

Complex conjugation of the second condition in (4.8) implies that

$$U^{-1}X_+U = \bar{\alpha}t_0 + \bar{\Phi}t_+. \tag{4.16}$$

Now apply

$$U^{-1}dU = U^*\tau^0 t_0 + \frac{1}{2}U^*\tau t_- + \frac{1}{2}U^*\bar{\tau}t_+ \tag{4.17}$$

to X_+, the condition in (4.16) is thus equivalent to

$$U^*\tau(X_+) = 0. \tag{4.18}$$

We now need to show that this is equivalent to U covering a holomorphic map.

Using the parameterisation of U in terms of the functions F_i defined in (4.10) we see that Equation (4.15) becomes

$$X_0\left(\frac{F_i}{\sqrt{|F_1|^2 - \lambda|F_2|^2}}\right) = \frac{i}{2}p\left(\frac{F_i}{\sqrt{|F_1|^2 - \lambda|F_2|^2}}\right). \tag{4.19}$$

From this it follows that the map $\pi \circ U = \frac{F_2}{F_1}$ has equivariant degree zero, from Definition 4.1, and that U covers

$$f = s^*\left(\frac{F_2}{F_1}\right) : M_0 \to M. \tag{4.20}$$

Using (4.2) for $\frac{F_2}{F_1}$ we find that f being holomorphic is equivalent to

$$X_+\left(\frac{F_2}{F_1}\right)=0. \tag{4.21}$$

Returning to (4.18) use (2.13) to see that

$$U^*\tau(X_+)=\frac{2i}{|F_1|^2-\lambda|F_2|^2}\left(F_1X_+F_2-F_2X_+F_1\right)=\frac{2i}{|F_1|^2-\lambda|F_2|^2}F_1^2X_+\frac{F_2}{F_1}, \tag{4.22}$$

with the last equality holding away from the zeros of F_1. Thus the condition (4.18) is equivalent to $f=s^*\left(\frac{F_2}{F_1}\right)$ being holomorphic away from the zeros of F_1. This means that for $\lambda\neq -1$ the result has been established. For the $\lambda=-1$ case f will have poles at the zeros of F_1 and is a holomorphic map $M_0\to S^2$. □

At the level of the bundle map U the $U(1)$ gauge invariance from (4.6) becomes

$$U\mapsto\tilde{U}=Ue^{\beta t_0},\qquad \beta\in C^\infty\left(\mathbb{H}^1_{\lambda_0}\right). \tag{4.23}$$

This defines a new trivialisation with the same f as U. The connection $\tilde{U}^{-1}d\tilde{U}$ differs from $\mathcal{A}=U^{-1}dU$ by the gauge transformation in (4.6).

Notice that when $\lambda\neq -1$ a vortex configuration can always be constructed from a given holomorphic map $f:M_0\to M$ by choosing

$$F_1(z_1,z_2)=1,\qquad F_2(z_1,z_2)=f\left(\frac{z_2}{z_1}\right). \tag{4.24}$$

This trivial lift results in a connection $\mathcal{A}$ that is constant along the fibres since $\mathcal{A}(X_0)=0$ which implies $\mathcal{L}_{X_0}\mathcal{A}=0$. This trivial lift is a direct consequence of $\mathbb{H}^1_1$ and $\mathbb{H}^1_0$ being trivial bundles. The trivial lift was observed for the $\lambda=\lambda_0=1$ case in [20].

4.2. Vortex configurations from vortices

To construct vortex configurations from a non-trivial lift we follow the work of [19,20] and lift vortices from M_0 to $\mathbb{H}^1_{\lambda_0}$. The idea behind this is that a vortex is given by a rational function $f=f_2/f_1$. We can then take the lift $f_2/f_1=F_2/F_1$. This lift is non-trivial since the functions F_1, F_2 have a non-trivial equivariant degree.

For this lift the Higgs field Φ has equivariant degree $2N-2$ in the sense of Definition 4.1, with N the integer equivariant degree of F_1 and F_2 determined from the rational function f. The specific vortex number depends on the type of vortex. Lifting Popov and hyperbolic vortices has been carried out previously in [19] and [20] respectively.

This leads us to the following Corollary of Theorem 4.3.

Corollary 4.4. *Let $U:\mathbb{H}^1_{\lambda_0}\to\mathbb{H}^1_\lambda$ be the bundle map from (4.10) with*

$$2iX_0F_1=2iX_0F_2=N\in\mathbb{Z}. \tag{4.25}$$

The vortex configuration (Φ, A) constructed from the connection $\mathcal{A}=U^{-1}dU$ through Theorem 4.3 has a gauge field which satisfies the normalisation condition

$$A(X_0)=N-1, \tag{4.26}$$

and a Higgs field of equivariant degree $2N-2$. In terms of F_1, F_2 the vortex configuration is expressed as

$$\Phi=\frac{F_1\partial_2F_2-F_2\partial_2F_1}{z_1\left(|F_1|^2+|F_2|^2\right)}, \tag{4.27}$$

and

$$A=(N-1)\,\sigma^0+\frac{i}{2}X_-\ln D^2\sigma-\frac{i}{2}X_+\ln D^2\bar{\sigma} \tag{4.28}$$

with $D^2=|F_1|^2+|F_2|^2$.

To make this result more understandable we give two example; one for Jackiw-Pi vortices, and another for Ambjørn-Olesen vortices.

4.2.1. Jackiw-Pi vortices

In [7] it was shown that the Jackiw-Pi vortex equations on $\mathbb{R}^2$ with a finite number of zeros are solved by a rational map $f = \frac{p}{q} : \mathbb{R}^2 \to S^2$ with $\deg(p) < \deg(q)$. For example a $2N$-vortex solution is given by

$$p(z) = \sum_{i=0}^{M} a_i z^i, \qquad q(z) = \sum_{i=0}^{N} b_i z^i, \quad M < N, \tag{4.29}$$

with the understanding that p and q have no common factors, at least one of a_0, b_0 are non-zero, and $b_N \neq 0$. In this case we can write down the following homogeneous polynomials

$$P(z_1, z_2) = \sum_{i=0}^{M} a_i z_1^{N-i} z_2^i, \qquad Q(z_1, z_2) = \sum_{i=0}^{N} b_i z_1^{N-i} z_2^i, \tag{4.30}$$

which satisfy

$$s^*\left(\frac{P}{Q}\right) = \frac{p}{q}. \tag{4.31}$$

Examples of Jackiw-Pi vortices with $N = 1$ and $N = 2$, including plots of $|\phi|^2$, are given in [6,8].

In Corollary 4.4 taking

$$F_1(z_1, z_2) = Q(z_1, z_2), \qquad F_2(z_1, z_2) = P(z_1, z_2), \tag{4.32}$$

with P, Q given in (4.30) so that $2iX_0 P = 2iX_0 Q = N$, defines a $\lambda_0 = 0, \lambda = -1$ vortex configuration.

In [13] the case of Jackiw-Pi vortices on the torus is discussed, there the map f is a doubly periodic elliptic function. As a vortex on the torus it has a finite vortex number, $2N$ where N is the number of poles of f. However, as a vortex on $\mathbb{R}^2$ it has an infinite number of zeros. The torus is obtained from $\mathbb{R}^2$ by quotienting with a discrete subgroup of SE_2 and demanding that the zeros of the Higgs field on $\mathbb{R}^2$ are periodic under this subgroup and there are $2N$ of them in the principal domain. The only way to lift these vortices seems to be via the trivial lift (4.24).

The most popular example of a Jackiw-Pi vortex on $\mathbb{R}^2$ [6–8] is the axially symmetric case constructed from the rational function

$$f = \frac{1}{z^N}. \tag{4.33}$$

For this choice of f the F_i are given by

$$F_1 = z_2^N, \qquad F_2 = z_1^N. \tag{4.34}$$

For this vortex the Higgs field of the vortex configuration is given by

$$\Phi = -N \frac{z_1^{N-1} z_2^{N-1}}{|z_1|^{2N} + |z_2|^{2N}}. \tag{4.35}$$

This can be explicitly seen to satisfy $2iX_0\Phi = (2N-2)\Phi$ and thus Φ is a degree $2N-2$ equivariant function.

4.2.2. Ambjørn-Olesen vortices

From [13] we know that Ambjørn-Olesen vortices are constructed from a holomorphic map $f : H^2 \to S^2$, subject to $|f(z)| \to 1$ as $|z| \to 1$. These maps can be expressed in terms of their m zeros, $c_1, \cdots, c_m$, and n poles, $d_1, \cdots, d_n$, as

$$f(z) = \frac{f_2}{f_1} = \prod_{i=1}^{m}\left(\frac{z - c_i}{1 - \bar{c}_i z}\right) \prod_{j=1}^{n}\left(\frac{1 - \bar{d}_j z}{z - d_j}\right). \tag{4.36}$$

To see this we use that the zeros and poles of f define the Blaschke products

$$f_2 = \prod_{i=1}^{m}\left(\frac{z - c_i}{1 - \bar{c}_i z}\right), \quad f_1 = \prod_{j=1}^{n}\left(\frac{z - d_j}{1 - \bar{d}_j z}\right). \tag{4.37}$$

The ratio of these Blaschke products $\frac{f_2}{f_1}$ has the same zeros and poles as f and their ratio $\frac{f f_1}{f_2}$ is a holomorphic function with no zeros and no poles satisfying $|f(z)| = 1$ for $|z| = 1$. Liouville's theorem then gives that this ratio is a constant, $\mu \in \mathbb{C}$ such that $|\mu| = 1$, multiplying f by a constant does not change the vortex that we construct from f so we can take $\mu = 1$.

To make use of Corollary 4.4 to construct a $\lambda_0 = -\lambda = 1$ vortex configuration take

$$F_1(z_1, z_2) = \prod_{i=1}^{n}\prod_{j=1}^{m} (z_1 - \bar{c}_i z_2)\left(z_2 - d_j z_1\right), \quad F_2(z_1, z_2) = \prod_{i=1}^{n}\prod_{j=1}^{m} (z_2 - c_i z_1)\left(z_1 - \bar{d}_j z_2\right). \tag{4.38}$$

The equivariant degree is $N = m + n$.

The same procedure can be used to construct vortex configurations from a given Bradlow vortex.

4.2.3. Lifts at the level of the connection

Working at the level of the flat connections $\mathcal{A}$ and $f^*\hat{A}$ we can state the relationship between vortex connections and vortices. To do this recall that if $\lambda = 1$ then $f^*\hat{A}$ has singularities at the points q_j such that $f_1(q_j) = 0$, and is thus defined on the space $P \subset \mathbb{H}^1_{\lambda_0}$ defined in Eq. (3.13).

Note that for a function $g : M_0 \to \mathbb{C}$ we can define the map

$$r_g : M_{\lambda_0}\backslash\{q_i\} \to \mathbb{H}^1_\lambda, \quad r_g = \begin{pmatrix} \frac{\bar{g}}{|g|} & 0 \\ 0 & \frac{g}{|g|} \end{pmatrix}, \tag{4.39}$$

where the q_i are the zeros of g.

Now using the section s defined in (2.24) we get the following corollary of Theorem 4.3 and Proposition 2.3.

Corollary 4.5. *For the bundle map U in (4.10) covering the holomorphic map f, the gauge vortex connection $f^*\hat{A}$ from (3.10) is related to $\mathcal{A} = U^{-1}dU$ through the, possibly singular, gauge transformation r_{f_1}, where $f_1 = F_1 \circ s$:*

$$f^*\hat{A} = r_{f_1}^{-1} s^*\mathcal{A} r_{f_1} + r_{f_1}^{-1} dr_{f_1}. \tag{4.40}$$

The trivial lift in (4.24) corresponds to $r_{f_1} = \mathbb{I}$, $f^*\hat{A} = s^*\mathcal{A}$. Again, this is only possible when $\lambda \neq 1$ so that $s^*\mathcal{A}$ and $f^*\hat{A}$ are both manifestly smooth.

5. A comment on magnetic modes

5.1. Group manifolds and stereographic projection

In the previous work [19,20], vortex configurations on the group manifold, either $SU(2)$ or $SU(1,1)$, were used to construct solutions to a twisted Dirac equation. These vortex magnetic modes were then pulled back to vortex magnetic modes on flat $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$. For both Bradlow and Ambjørn-Olesen vortices the approach used in [20] is applicable and the vortices lead to solutions of a twisted Dirac equation on $\mathbb{R}^{2,1}$. For completeness we give the full argument here and stress that the results of this section are all under the assumption that $\lambda_0 \neq 0$. This is because both Jackiw-Pi and Laplace vortices, which have $\lambda_0 = 0$, are related to the group SE_2 which does not possess a bi-invariant metric. In fact the metric from Eq. (2.6) is singular[5] so we cannot construct a Dirac operator in the usual way. A potential approach to fixing this problem is to centrally extend SE_2 to the Nappi-Witten space [16] which has a Lorentzian metric. This central extension would not affect the construction in the other cases and the hope is that it will enable the construction of vortex magnetic modes from Jackiw-Pi vortices. This is a current direction of research.

The Killing metric on the Lie algebra of $\mathbb{H}^1_{\lambda_0}$ depends on λ_0 and is

$$ds^2_{\lambda_0} = -\frac{1}{\lambda_0}\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2, \tag{5.1}$$

in particular this metric would be singular if the $\lambda_0 = 0$ case was included. We are assuming the oriented (pseudo) orthonormal co-frame

$$\left(dx^0, dx^1, dx^2\right), \tag{5.2}$$

such that the volume form is

$$dx^0 \wedge dx^1 \wedge dx^2. \tag{5.3}$$

[5] Eq. (2.6) says that the inverse metric is degenerate which is equivalent to the metric being singular.

A point $\vec{x} \in \mathbb{R}^3_{\lambda_0}$ is given by $\vec{x} = x^a \vec{b}_a$ with $\vec{b}_a$ an oriented basis such that $g(\vec{b}_a, \vec{b}_b) = g_{ab}$, for g as in (2.6). For $\vec{x}, \vec{y} \in \mathbb{R}^3_{\lambda_0}$ the scalar product is given by

$$\vec{x} \cdot \vec{y} = g_{ab} x^a y^b, \tag{5.4}$$

and the distance to $\vec{x}$ is

$$r^2 = \vec{x} \cdot \vec{x} = -\lambda_0 (x_0)^2 + (x_1)^2 + (x_2)^2. \tag{5.5}$$

The cross product of $\vec{x}, \vec{y} \in \mathbb{R}^3_{\lambda_0}$ is

$$\vec{x} \times \vec{y} = \varepsilon_{ij}{}^k x^i y^j b_k, \tag{5.6}$$

with $\varepsilon_{012} = 1$, $\varepsilon^{012} = -\lambda_0$.

The Hodge star on the basis is computed in the standard way as

$$\star \left(dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) = \frac{1}{(3-k)!} \varepsilon^{i_1, \cdots, i_k}{}_{i_{k+1}, \cdots, i_3} dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_3}. \tag{5.7}$$

We call this space

$$\mathbb{R}^3_{\lambda_0} = \left(\mathbb{R}^3, ds^2_{\lambda_0} \right). \tag{5.8}$$

On $\mathbb{H}^1_{\lambda_0}$ we work with the oriented, orthonormal co-frame

$$\left(\frac{1}{2}\sigma^0, \frac{1}{2}\sigma^1, \frac{1}{2}\sigma^2 \right), \tag{5.9}$$

with respect to which the bi-invariant metric and orientation are given by Eqs. (2.14) and (2.15) respectively. The slightly unconventional ordering is so that this gives the volume form on $\mathbb{R}^3$ after stereographic projection in the $\lambda = 1, -1$ cases.

We construct two maps between $\mathbb{R}^3_{\lambda_0}$ and $\mathbb{H}^1_{\lambda_0}$,

$$G, H : \mathbb{R}^3_{\lambda_0} \to \mathbb{H}^1_{\lambda_0}. \tag{5.10}$$

As in the earlier papers [19,20], H is a scaled version of inverse stereographic projection and G is the inverse gnomonic projection. The two maps are related through $H(\vec{x}) = G(\vec{x})^2$. Note that, as in [20], G, H are not maps from all of $\mathbb{R}^3_{\lambda_0}$ to $\mathbb{H}^1_{\lambda_0}$ but only from the subspace $\mathcal{I} \subset \mathbb{R}^3_{\lambda_0}$ defined as

$$\mathcal{I} = \left\{ \left(x^0, x^1, x^2 \right) \in \mathbb{R}^3_{\lambda_0} | \lambda_0 r^2 < 1 \right\}. \tag{5.11}$$

In the $SU(2)$ ($\lambda_0 = -1$) case, $\mathcal{I} = \mathbb{R}^3$ since the above condition becomes $r^2 > -1$. However, in the $SU(1,1)$ ($\lambda_0 = 1$) case, the condition on r^2 is that

$$r^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 < 1, \tag{5.12}$$

and $\mathcal{I}$ is the interior of a single sheeted hyperboloid.

In the notation used here H is

$$H : \mathcal{I} \to \mathbb{H}^1_{\lambda_0}, \tag{5.13}$$

$$\vec{x} \mapsto \frac{1+\lambda_0 r^2}{1-\lambda_0 r^2} \mathbb{I} - \frac{4}{1-\lambda_0 r^2} \vec{x} \cdot \vec{t} = \frac{1}{1-\lambda_0 r^2} \begin{pmatrix} 1+\lambda_0 r^2 + 2ix^0 & -2i\lambda_0 \left(x^1 - ix^2 \right) \\ 2i \left(x^1 + ix^2 \right) & 1+\lambda_0 r^2 - 2ix^0 \end{pmatrix}. \tag{5.14}$$

The analogue of the inverse gnomonic projection is

$$G : \mathcal{I} \to \mathbb{H}^1_{\lambda_0}, \qquad \vec{x} \mapsto \frac{\mathbb{I} - 2\vec{x} \cdot \vec{t}}{\sqrt{1-\lambda_0 r^2}}. \tag{5.15}$$

Pulling back the left-invariant one-forms, σ^a, with H we define a co-frame on $\mathbb{R}^3_{\lambda_0}$, ϑ^a, as

$$H^* \sigma^a = -\frac{1}{\Omega} \vartheta^a, \tag{5.16}$$

where the scale factor Ω is defined as

$$\Omega = \frac{1-\lambda_0 r^2}{4}. \tag{5.17}$$

The co-frame ϑ^a is related to the standard flat co-frame through conjugation by G,

$$\begin{aligned}\vec{\vartheta}\cdot\vec{t} &= \frac{1}{1-\lambda_0 r^2}\left(\left(1+\lambda_0 r^2\right)(d\vec{x}\cdot\vec{t}) - 2\lambda_0\left(d\vec{x}\cdot\vec{x}\right)\left(\vec{x}\cdot\vec{t}\right) + 2\left(\vec{x}\times d\vec{x}\right)\cdot\vec{t}\right),\\ &= G^{-1}\left(d\vec{x}\cdot\vec{t}\right)G.\end{aligned} \tag{5.18}$$

In other words, the ϑ^a are a rotated basis for the co-tangent space, with the rotation given by G acting in the adjoint representation.

Lemma 5.1. *The pull-backs of the Maurer-Cartan one-form on $\mathbb{H}^1_{\lambda_0}$ by the maps G, H are related via*

$$H^{-1}dH = G^{-1}dG + G^{-1}(G^{-1}dG)G, \tag{5.19}$$

with the inverse relation

$$G^{-1}dG = \frac{1}{2}H^{-1}dH - \lambda_0 \star (d\Omega \wedge H^{-1}dH). \tag{5.20}$$

Here $\star$ is the Hodge star operator on $\mathbb{H}^1_\lambda$ with respect to the orientation (2.15).

Proof. This result follows from a direct computation. The first formula is found by substituting $H = G^2$ into $H^{-1}dH$. For the second formula use Eq. (5.19) to rewrite Eq. (5.20) as

$$-2\lambda_0 \star \left(d\Omega \wedge \left(dGG^{-1} + G^{-1}dG\right)\right) = dGG^{-1} - G^{-1}dG. \tag{5.21}$$

Next computing

$$G^{-1}dG = -2\frac{d\vec{x}\cdot\vec{t} + \left(\vec{x}\times d\vec{x}\right)\cdot\vec{t}}{1-\lambda_0 r^2}, \tag{5.22}$$

leads to

$$dGG^{-1} + G^{-1}dG = -4\frac{\left(d\vec{x}\cdot\vec{t}\right)}{\left(1-\lambda_0 r^2\right)}, \quad dGG^{-1} - G^{-1}dG = 4\frac{\left(\vec{x}\times d\vec{x}\right)\cdot\vec{t}}{\left(1-\lambda_0 r^2\right)}. \tag{5.23}$$

Using these expressions along with

$$-2\lambda_0 d\Omega = \lambda_0^2 \vec{x}\cdot d\vec{x} = \vec{x}\cdot d\vec{x}, \tag{5.24}$$

since $\lambda_0 = \pm 1$, and

$$\star\left(\vec{x}\cdot d\vec{x} \wedge d\vec{x}\cdot\vec{t}\right) = \left(d\vec{x}\times\vec{x}\right)\cdot\vec{t}, \tag{5.25}$$

gives (5.21) and thus (5.20) is established. □

5.2. Dirac operators

Next construct the Dirac operators on both $\mathbb{H}^1_{\lambda_0}$ and $\mathbb{R}^3_{\lambda_0}$. The spin connection on $\mathbb{H}^1_{\lambda_0}$ is

$$\Gamma_{\mathbb{H}^1_{\lambda_0}} = -\frac{1}{8}[\gamma_a, \gamma_b]\omega^{ab} = \frac{1}{2}h^{-1}dh. \tag{5.26}$$

Introducing θ such that $\theta^2 = \lambda_0$, this is 1 in the Lorentzian case and i in the Euclidean case, the Dirac operator is written as

$$\not{D}_{\mathbb{H}^1_{\lambda_0}} = 4i\theta t^a X_a + \frac{3}{2}i\lambda_0\theta\mathbb{I}, \tag{5.27}$$

$$= -2\theta\begin{pmatrix}\lambda_0 X_0 & \lambda_0 X_- \\ -X_+ & -\lambda_0 X_0\end{pmatrix} + \frac{3}{2}i\lambda_0\theta\mathbb{I}. \tag{5.28}$$

The Dirac operator can be minimally coupled to an Abelian gauge potential A as

$$\begin{aligned}\not{D}_{\mathbb{H}^1_{\lambda_0},A} &= 4i\theta t^a\left(X_a + iA_a\right) + \frac{3}{2}i\lambda_0\theta\mathbb{I}\\ &= -2\theta\begin{pmatrix}\lambda_0\left(X_0 + iA_0\right) & \lambda_0\left(X_- + iA_-\right) \\ -\left(X_+ + iA_+\right) & -\lambda_0\left(X_0 + iA_0\right)\end{pmatrix} + \frac{3}{2}i\lambda_0\theta\mathbb{I}.\end{aligned} \tag{5.29}$$

The Dirac operator on $\mathbb{R}^3_{\lambda_0}$ minimally coupled to the gauge potential $\vec{A}\cdot\vec{dx}$ is

$$\not{D}_{\mathbb{R}^3_{\lambda_0},A} = 2i\theta t^a\left(\partial_a + iA_a\right) \tag{5.30}$$

Definition 5.2. A spinor $\Psi : \mathbb{H}^1_{\lambda_0} \to \mathbb{C}^2$ that satisfies

$$\not{D}_{\mathbb{H}^1_{\lambda_0},A}\Psi = 0 \tag{5.31}$$

is called a magnetic mode or magnetic Dirac mode of the Dirac operator $\not{D}_{\mathbb{H}^1_{\lambda_0}}$, coupled to the connection A.

Lemma 5.3. *If $\Psi : \mathbb{H}^1_{\lambda_0} \to \mathbb{C}^2$ is a magnetic Dirac mode of the Dirac operator (5.29) on $\mathbb{H}^1_{\lambda_0}$ coupled to the $U(1)$ gauge field A, then*

$$\Psi_H = G\Omega^{-1}H^*\Psi \tag{5.32}$$

is a magnetic mode of the Dirac operator $\not{D}_{\mathbb{R}^3_{\lambda_0},H^*A}$ on Euclidean 3-space coupled to the connection H^*A.

Proof. This result follows from the known equivariance properties of the Dirac operator under scaling and changes of frame. Here we give an explicit verification in the interest of providing a complete discussion. Consider the pull-back of the spin connection

$$H^*\Gamma_{\mathbb{H}^1_{\lambda_0}} = \frac{1}{2}H^{-1}dH. \tag{5.33}$$

Then (5.19) implies that

$$d + \frac{1}{2}H^{-1}dH = \Omega G^{-1}\left(d + \frac{1}{2}\left(GdG^{-1} + G^{-1}dG\right) + \Omega^{-1}d\Omega\right)\Omega^{-1}G. \tag{5.34}$$

Next using that $GdG^{-1} = -dGG^{-1}$, (5.23), and

$$\Omega^{-1}d\Omega = -2\lambda_0\frac{\vec{x}\cdot\vec{dx}}{1-\lambda_0 r^2}, \tag{5.35}$$

leads to

$$t^a i_{\partial_a}\left(\frac{1}{2}\left(GdG^{-1} + G^{-1}dG\right) + \Omega^{-1}d\Omega\right) = \frac{2\lambda_0\vec{x}\cdot\vec{t}}{1-\lambda_0 r^2} - \frac{2\lambda_0\vec{x}\cdot\vec{t}}{1-\lambda_0 r^2} = 0. \tag{5.36}$$

Putting everything together, the pull-back of the Dirac operator on $\mathbb{H}^1_{\lambda_0}$ with spin connection $\Gamma_{\mathbb{H}^1_{\lambda_0}}$ and coupled to the abelian connection A is

$$\frac{1}{4i\theta}\not{D}_{\mathbb{H}^1_{\lambda_0},A} = t^a i_{H^*X_a}\left(d + \frac{1}{2}H^{-1}dH + iH^*A\right) \tag{5.37}$$

$$= -\Omega G^{-1}t^a i_{\partial_a}G\left(d + \frac{1}{2}H^{-1}dH + iH^*A\right) \tag{5.38}$$

$$= -\Omega^2 G^{-1}t^a i_{\partial_a}\left(d + iH^*A\right)\Omega^{-1}G, \tag{5.39}$$

giving the stated relationship between magnetic modes of $\not{D}_{\mathbb{H}^1_{\lambda_0},A}$ and $\not{D}_{\mathbb{R}^3_{\lambda_0},H^*A}$. □

5.3. Vortex magnetic modes

As observed in [19,20] vortex configurations give rise to magnetic modes satisfying a second non-linear equation. The results presented here unify the earlier results and extend them to include a construction of vortex magnetic modes from Bradlow and Ambjørn-Olesen vortex configurations.

Definition 5.4. A pair (Ψ, A) of a spinor Ψ and a one-form A on $\mathbb{H}^1_{\lambda_0}$ is said to be a vortex magnetic mode of the Dirac equation on $\mathbb{H}^1_{\lambda_0}$ if

$$\not{D}_{\mathbb{H}^1_{\lambda_0},A}\Psi = 0, \quad F_A = -\frac{\lambda}{\lambda_0}4i\star\Psi^\dagger h^{-1}dh\Psi - \lambda_0\frac{1}{4}\sigma^1\wedge\sigma^2, \tag{5.40}$$

with $\star$ the Hodge star operator on $SU(1,1)$ with respect to the metric (2.14) and orientation (2.15).

Theorem 5.5. *Given a vortex configuration* (Φ, A) *on* $\mathbb{H}^1_{\lambda_0}$ *the pair*

$$\Psi = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}, \qquad A' = A + \frac{3}{4}\sigma^0, \tag{5.41}$$

is a vortex magnetic mode on $\mathbb{H}^1_{\lambda_0}$.

Proof. As (Φ, A) is a vortex configuration it solves

$$X_0\Phi + iA_0\Phi = 0, \qquad X_+\Phi + iA_+\Phi = 0, \tag{5.42}$$

these are just the contraction of (4.3) with (X_0, X_-) and (X_+, X_-) respectively. Now, the spinor in the Theorem is a magnetic mode if

$$X_0\Phi + iA_0'\Phi - \frac{3i}{4}\Phi = 0, \qquad X_+\Phi + iA_+'\Phi = 0. \tag{5.43}$$

Since, $A_0' = A'(X_0) = A_0 + \frac{3}{4}$ and $A'(X_+) = A(X_+)$ these equations follow from (5.42).

For the non-linear equation consider that for the given form of spinor we know that

$$-\frac{\lambda}{\lambda_0}4i \star \Psi^\dagger h^{-1} dh \Psi = -\frac{\lambda}{\lambda_0}4i|\Phi|^2 \star \left(-\frac{i}{2}\sigma^0\right) = \lambda|\Phi|^2\sigma^2 \wedge \sigma^1. \tag{5.44}$$

On the other hand

$$F_{A'} = F_A - \frac{3}{4}\lambda_0\sigma^2 \wedge \sigma^1 = \left(\lambda|\Phi|^2 - \frac{\lambda_0}{4}\right)\sigma^1 \wedge \sigma^2, \tag{5.45}$$

which is nothing but the non-linear equation from Eq. (5.40). □

5.4. Vortex magnetic modes on flat space

Combining Theorem 5.5 with Lemma 5.3, vortex magnetic modes on $\mathbb{H}^1_{\lambda_0}$ can be converted to magnetic modes on $\mathbb{R}^3_{\lambda_0}$.

Before stating what vortex magnetic modes pull back to on $\mathbb{R}^3_{\lambda_0}$ we need to establish what happens to the inhomogeneous term in (5.40). Computing its pullback we find

$$\frac{1}{4}H^*\left(\sigma^1 \wedge \sigma^2\right) = \frac{4}{(1-\lambda_0 r^2)^2} \star_{\mathbb{R}^3_{\lambda_0}} \vartheta^0 = \frac{1}{2}\varepsilon^a{}_{bc} b_a dx^b \wedge dx^c. \tag{5.46}$$

The corresponding magnetic field

$$\vec{b} = \frac{1}{1-\lambda_0 r^2}\begin{pmatrix} 1 + \lambda_0 r^2 + 2x_0^2 \\ 2\left(\lambda_0 x_2 + x_1 x_0\right) \\ -2\left(\lambda_0 x_1 - x_2 x_0\right) \end{pmatrix}, \tag{5.47}$$

is a background magnetic field, with field lines the fibres of the fibration $\pi : \mathbb{H}^1_{\lambda_0} \to M_{\lambda_0}$. In the $\lambda = \lambda_0 = -1$ case considered in [19] the field lines are the fibres of the Hopf fibration and thus are all linked.

In the $\lambda_0 = 1$ case the only differences between the three types of vortex magnetic modes on $\mathbb{R}^3_1 = \mathbb{R}^{1,2}$ comes from the different coefficients of $\Psi^\dagger h^{-1} dh\Psi$ in (5.40). More precisely, vortex magnetic modes constructed from the Hyperbolic, Bradlow, and Ambjørn-Olesen vortices, differ only in the relationship between $\Psi^\dagger h^{-1} dh\Psi$ and F_A'.

Definition 5.6. A pair (Ψ, A) of a spinor Ψ and a one-form $A = \vec{A}\cdot d\vec{x}$ is called a vortex magnetic mode on $\mathbb{R}^3_{\lambda_0}$ if the coupled equations

$$\not{D}_{\mathbb{R}^3_{\lambda_0}, H^*A}\Psi = 0, \qquad \vec{B} = -2i\frac{\lambda}{\lambda_0}\Psi_H^\dagger \vec{t}\Psi_H - \lambda_0\vec{b}, \tag{5.48}$$

where $\vec{B} = \nabla \times \vec{A}$ and $\vec{b}$ is the background field (5.47), are satisfied.

The means of constructing examples of such vortex magnetic modes are through the following Corollary of our earlier results.

This starts from a bundle map and results in explicit expressions for the spinor and gauge potential in terms of a vortex configuration.

Corollary 5.7. *A bundle map $U : \mathbb{H}^1_{\lambda_0} \to \mathbb{H}^1_\lambda$ covering a holomorphic map $f : M_{\lambda_0} \to M_\lambda$ determines a smooth vortex magnetic mode on $\mathcal{I} \subset \mathbb{R}^3_{\lambda_0}$. In terms of the vortex configuration (Φ, A) extracted from $U^{-1}dU$ using (4.7) the vortex magnetic mode is*

$$\Psi_H = \Omega^{-1} G \begin{pmatrix} H^*\Phi \\ 0 \end{pmatrix}, \qquad A' = H^*\left(A + \frac{3}{4}\sigma^0\right). \tag{5.49}$$

The proof of this result follows by bringing together the earlier results. Theorem 4.3 gives the construction of a vortex configuration from a bundle map, then Theorem 5.5 shows how vortex configurations give rise to vortex magnetic modes, finally Lemma 5.3 tells us how to turn magnetic modes on $\mathbb{H}^1_{\lambda_0}$ into magnetic modes on $\mathbb{R}^3_{\lambda_0}$. The only thing that needs to be checked is that the non-linear equation in Eq. (5.48) is satisfied. This follows from a straightforward computation.

6. Vortices and instantons

The representation of vortices as non-abelian connections in three dimensions given here is an alternative to viewing vortices as symmetric, non-abelian instantons on flat $\mathbb{R}^4$. In [3] it was shown that all five of the integrable vortex equations can be constructed as the dimensional reduction of an appropriate (anti-)self dual Yang-Mills theory on $M_0 \times N_0$, where N_0 is a Riemann surface with constant Gauss curvature $-K_0$. This construction is based on the general story of gauge fields which possess space time symmetries introduced in [4]. In [19,20] it was observed that there is an interesting relationship between the gauge group of the Yang-Mills theory and the Lie algebra that the Cartan connection is valued in. In short if the Cartan group is $\mathbb{H}^1_\lambda$ then the instanton gauge group is $\mathbb{H}^1_{-\lambda}$.

In our conventions the construction from [3] considers instantons on the four manifold $\mathbb{R}^4 \simeq M_0 \times N_0$ with gauge group $\mathbb{H}^1_{-\lambda}$ that are equivariant with respect to the action of $\mathbb{H}^1_{-\lambda_0}$. This amounts to the instanton being independent of the N_0 factor and thus reducing to a (λ_0, λ) vortex on M_{λ_0}. Explicitly the instanton is given by the $\mathbb{H}^1_{-\lambda}$-connection

$$\mathcal{A}_{\mathrm{CD}} = -\left(a - \Gamma_{N_0}\right) t_0 + \frac{i\phi}{2}\bar{e}_{N_0} t_- - \frac{i\bar{\phi}}{2} e_{-N_0} t_+, \tag{6.1}$$

with (ϕ, a) the (λ_0, λ) vortex on M_0 and e_{N_0}, Γ_{N_0} the complexified co-frame and spin connection on N_0. From Corollary 3.2 a (λ_0, λ) vortex is equivalent to a flat $\mathbb{H}^1_\lambda$-connection

$$A = -\left(a + \Gamma_0\right) t_0 + \frac{i\phi}{2} e_0 t_- - \frac{i\bar{\phi}}{2}\bar{e}_0 t_+. \tag{6.2}$$

It is interesting to contrast the two connections. $\mathcal{A}_{\mathrm{CD}}$ is an anti-self dual connection on a conformally flat four-manifold while A is a flat connection on a Riemann surface. This manifests itself in the fact that A only depends on information on M_0, the vortex (ϕ, a) and the co-frame field and spin connection. On the other hand $\mathcal{A}_{\mathrm{CD}}$ depends on information from both M_0 and N_0. Another difference that should be noted is that while we have used the same notation for the generators, $t_0, t_\pm$ they are not exactly the same, the key difference is in t_+. For $\mathbb{H}^1_\lambda$ the explicit form of t_+ is

$$t_+ = \begin{pmatrix} 0 & i\lambda \\ 0 & 0 \end{pmatrix}. \tag{6.3}$$

This means the sign in t_+ is different for the two connections.

However, we know that the flat connection in two dimensions is related to a flat connection on the group manifold $\mathbb{H}^1_{\lambda_0}$ given by Eq. (4.7). An immediate question is if there is a way to go directly between the instanton and the three dimensional Cartan connection. At the moment we only know how to pass between them by going through the vortex in two dimensions. There are definitely key differences in their construction with A being constructed from the pullback of the left-invariant Maurer-Cartan one-form on $\mathbb{H}^1_\lambda$ while $\mathcal{A}_{\mathrm{CD}}$, following the general construction in [4], is constructed from left-invariant data from $\mathbb{H}^1_{-\lambda_0}$ and right-invariant data from $\mathbb{H}^1_{-\lambda}$.

Finally consider the diagram

$$\begin{array}{ccc} \mathbb{H}^1_{-\lambda_0} \times \mathbb{H}^1_{\lambda_0} & \xrightarrow{U} & \mathbb{H}^1_\lambda \times \mathbb{H}^1_{-\lambda} \\ \downarrow{\pi} & & \downarrow{\pi} \\ N_0 \times M_0 & \xrightarrow{f} & M \times N \end{array} \tag{6.4}$$

where $f : M_0 \to M$ is the rational map defining a vortex and $U : \mathbb{H}^1_{\lambda_0} \to \mathbb{H}^1_\lambda$ is the bundle map encountered in Theorem 4.3. From the instanton point of view $\mathbb{H}^1_{-\lambda_0}$ would be the symmetry group and $\mathbb{H}^1_{-\lambda}$ is the gauge group. This could be flipped round to

$$\begin{array}{ccc} \mathbb{H}^1_{\lambda_0} \times \mathbb{H}^1_{-\lambda_0} & \xleftarrow[V]{} & \mathbb{H}^1_{-\lambda} \times \mathbb{H}^1_{\lambda} \\ \downarrow \pi & & \downarrow \pi \\ M_0 \times N_0 & \xleftarrow[g]{} & N \times M \end{array} \tag{6.5}$$

with $g : N \to N_0$ a rational map defining a vortex and $V : \mathbb{H}^1_{-\lambda} \to \mathbb{H}^1_{-\lambda_0}$ a bundle map. Now the instanton point of view has $\mathbb{H}^1_{\lambda}$ as the symmetry group and $\mathbb{H}^1_{\lambda_0}$ as the gauge group.

This suggests that at the level of the groups there is a potential duality between the different vortex equations. This duality takes the (λ_0, λ) vortex equations to the $(-\lambda_0, -\lambda)$ vortex equations.

The Hyperbolic and Popov vortex equations are exchanged under this, as are the Bradlow and Jackiw-Pi vortices while both the Ambjørn-Olesen and Laplace vortex equations are "self-dual" in this sense.

7. Conclusions and outlook

This paper has considered the general problem of giving a geometric description of integrable abelian vortices as non-abelian flat connections. This provides a unified three dimensional interpretation of vortices, complementing the two dimensional metric geometry interpretation given by Baptista [2], and the four dimensional description of vortices as symmetric instantons [3].

The story is summarised in Fig. 1 where the most important maps, spaces and equations are given. This gives a unifying picture, generalising the work of [19,20] to include all of the integrable vortex equations considered in [13]. As well as establishing the relationship between vortices and Cartan geometry we have also discussed proposals to construct solutions to massless Dirac equations from vortices.

A comparison between the Cartan connection picture and the instanton picture of vortices leads to some intriguing comparisons. Not least the fact that there seems to be a duality at the level of the groups where the (λ_0, λ) vortex equations are sent to the $(-\lambda_0, -\lambda)$ vortex equations.

Recently the story of unified vortex equations has been extended. In [22] non-abelian analogues of the integrable abelian vortex equations have been considered. While in [5], magnetic defects were added which preserved the integrability of the abelian vortex equations. It is unknown if in either of these cases there is still a geometric understanding of the vortices, either in the Baptista sense, or in the Cartan geometry sense of this paper. Understanding these cases is a direction worth pursuing.

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