On the Optimal Construction of the Human Develpoment Index

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Abstract

We consider the optimality of the weighting scheme used to construct the Human Development Index (HDI) using an approach that relies on consistent tests for stochastic dominance efficiency. We consider consistent tests for stochastic dominance efficiency of a given index portfolio with respect to all possible indices constructed from a set of individual components. The test statistics and the estimators are computed using mixed integer programming methods. The results show that the equally weighted (fixed weights) HDI index is not optimal and that education by itself offers the best (optimal) indicator of welfare.

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1 Introduction

There exists a large literature that examines the distribution of certain attributes such as income across countries using country level average data such as per capital GDP taken as a proxy for the quality of life of individuals living in different countries. However, the use of per capita income to evaluate welfare improvements assumes that it reflects the level of economic welfare enjoyed by the average person, see Becker, Philipson and Soares (2005). It is also widely acknowledged that national income constitutes an imperfect measure of social well-being, see Easterlin (1995). For example, national income includes expenditures affecting social well-being in a negative way ("regrettable necessities") and ignores also the numerous components of well-being such as the enjoyment of good health, of an unpolluted natural environment, of leisure time and of political freedoms and rights, see Ponthiere (2004). Furthermore, as argued by Quah (1996), it is not reasonable to give the same weight to populous countries such as China or India and to countries with small populations. Another problem with using country level per capita averages is that it is assumed implicitly that each person in a country receives the same level of the attribute in question (in this case income). Given the uneven distribution of most attributes within each country, this practice causes substantial biases in the estimation of global welfare indices that depend on the distribution of the attributes, such as inequality or poverty indices. The past decade has seen some work that tried to estimate the distribution of some attributes of the world population. Different weights are given to different countries. More importantly, the distributions within each country are also considered by using summary statistics of various quantiles with each country. For example, see Bourguignon and Morrisson (2002) and Sala-I-Martin (2006) on world income distribution, and Bourguignon and Morrisson (2002) and Pradhan et al. (2003) on world health distribution.

A recent development in welfare economics is the increasing emphasis on multivariate analysis, see Maasoumi (1999) for an overview of multidimensional welfare analysis. It has been recognized that welfare analysis based on a single attribute is inadequate. An alternative basic needs approach contends that individual well-being and social welfare depends on the joint distribution of various attributes, such as income, health and education. Traditionally, the welfare analysis of multiple attributes are often undertaken by examining each individual attribute separately. Obviously, this approach fails to account for any relationships between various attributes. Alternatively, another popular method is to construct a single welfare index as an aggregate of multiple indices by single attributes. In this category we have the United Nation's Human Development Index (HDI), which is the arithmetic average of an income index, an education index and a health index and different versions of and the Human Poverty Index (HPI).

The HDI is a summary composite index that measures a countries average achievements in three basic aspects of human development: longevity, knowledge and a decent standard of living *using fixed equal weights.* Longevity is measured by life expectancy at birth; knowledge is measured by a combination of the adult literacy rate and the combined primary, secondary and tertiary gross enrollment ratio and standard of living by GDP per capita. The HPI is a measure that attempts to capture the many dimensions of poverty that exist in both poor and rich countries relying on fixed equal weights as well. The HPI-1-human poverty index for developing countries-measures human deprivations in the same three categories as HDI (longevity, knowledge and a decent standard of living). HPI-2 -human poverty index for selected high-income OECD countries-includes in addition to the three dimensions in HPI-1, social exclusion. A serious shortcoming is that the construction of all of the above hybrid measures as in the case of the separate analysis of single attributes, ignores the association among the various attributes. For example, suppose there exists a simple economy with two individuals each endowed with two attributes X and Y. Consider two scenarios. The first one is $[X_1 = 2, X_2 = 1]$ and $[Y_1 = 1, Y_2 = 2]$, while the second one $[X_1 = 2, X_2 = 1]$ and $[Y_1 = 2, Y_2 = 1]$. Since the marginal distributions of X and Y are identical between the two scenarios, any "hybrid" index that fails to accounts for the dependence between two marginal distributions will conclude that the two scenarios share the same level of welfare.

In order to examine the optimality of the fixed equal weighting scheme in the construction of the HDI Index, we first examine the stochastic dominance of the HDI index over a twenty five year period and determine which factors drive its improvement over time. In order to look for stochastic dominance over time, we rely on Kolmogorov-Smirnov type tests developed within a consistent testing environment developed by Barrett and Donald (2003), hereafter BD. We use this framework to test for stochastic dominance of HDI and its individual components over a 25 year period. This offers a generalization to Anderson (1996), Beach and Davidson (1983), Davidson and Duclos (2000) who have looked at second order stochastic dominance using tests that rely on pair-wise comparisons made at a fixed number of arbitrary chosen points. This is not a desirable feature since it introduces the possibility of test inconsistency. Davidson and Duclos (2000) have discussed the importance of first, second and third order stochastic dominance concepts (SD1, SD2, and SD3 respectively) between income distributions for social welfare and poverty rankings of distributions. Ravallion (1994) called the SD1 a poverty incidence curve, that of SD2 a poverty deficit curve, and that of SD3 a poverty severity curve. In an important similar application to optimal portfolio construction, Scaillet and Topaloglou (2006), hereafter ST, use stochastic dominance efficiency tests that can compare a given portfolio with an optimal diversified portfolio constructed from a set of assets. In our case we follow the same methodology, using the set of attributes (in our case per capita income, life expectancy and a measure of human capital) to construct the optimal hybrid index, that *does not* rely on fixed weights as HDI does.

The remainder of the paper as follows. In section 2, we examine the main framework of analysis, we define the notions of stochastic dominance efficiency and we discuss the general hypothesis for stochastic dominance at any order. We follow BD to describe the test statistics and its asymptotic properties. Following BD and ST, in section 3, we describe two practical ways to compute p-values for testing stochastic dominance at any order by looking at bootstrap methods and discuss the theoretical justification for the methods. Section 4 looks at the data and offers descriptive statistics, whereas in Section 5, we examine the empirical application of these methods on HDI and its components – education, longevity and standard of living – over twenty five years (1975, 1980, 1985, 1990, 1995 and 2000). In section 6, we use the ST methodology to ascertain whether the HDI index is optimally constructed using equal weights, or whether we can obtain an alternative portfolio with optimal weights for the different constituent components of the index. We provide the empirical application of ST tests and give the optimal portfolio for HDI index. Section 7 concludes and the appendix describes some of the details of the BD approach.

2 Hypothesis, Test Statistic and Asymptotic Properties

2.1 Stochastic Dominance and Hypothesis Formulation

We focus on a situation in which we have independent samples of indices from two populations that have associated cumulative distribution functions ($CDFs$) given by G and F. First order stochastic dominance (hereafter SD1) of G over F corresponds to $G(z) \leq F(z)$ for all z. When this occurs social welfare in the population summarized by G is at least as large as that in the F population for any social welfare function of the form $W(H) = \int U(z)dH(z)$ where H is the distribution of HDI index or its components (education index, longevity index and GDP index) and U is any increasing monotonic function of $z - i.e.$ $U'(z) \geq 0$.The CDF of distribution F is always at least as large as that of distribution G, $i.e.$ distribution F always has more mass in the lower part of distribution.

How is this related to HDI index dominance? Suppose we have n countries in total. If the CDF of HDI in 1975, $F(z)$, is always at least as large as that of the CDF in 1985, $G(z)$ at any point, then the proportion of countries below the particular index level, for year 1975 is higher than that of 1985. Therefore, the 1985 HDI index stochastically dominates its 1975 counterpart in the first order sense.

When the two CDF curves intersect, then the ranking is ambiguous. In this situation we can not state whether one distribution first order dominates the other. This leads to an ambiguous situation which makes it necessary to use higher order stochastic dominance.

Second order stochastic dominance ($SD2$) of G over F corresponds to $\int_0^z G(t)dt \leq \int_0^z F(t)dt$ for all z and the social welfare in the population summarized by G is at least as large as that in the F population for any social welfare function of the form $W(H)$, where U is monotonically increasing and concave-that is $U'(z) \geq 0$ and $U''(z) \leq 0$. Second order stochastic dominance is verified, not by comparing the $CDFs$ themselves, but comparing the integrals below them. We examine the area below the $F(z)$ and $G(z)$ curves. Given lower and upper boundary levels, we determine the area beneath the curves and, if the area beneath the $F(z)$ distribution is larger than the one of $G(z)$, then in this case $G(z)$ stochastically dominates $F(z)$ in second order sense. Since we look at the area under the distributions, second order dominance implies simply an overall improvement and not a point-wise dominance over all the points of the support of one distribution over another.

There is no guarantee that the second order dominance will hold, so one may want to look for third order dominance. Third order stochastic dominance ($SD3$) of G over F corresponds to $\int_0^z \int_0^s G(t)dtds \leq \int_0^z \int_0^s F(t)dtds$ for all z and the social welfare in the population summarized by G is at least as large as that in the F population for any social welfare function of the form $W(H)$ where U satisfies $U'(z) \geq 0$, $U''(z) \leq 0$, and $U'''(z) \geq 0$. This is the case of third-order stochastic dominance and it is equivalent to imposing the condition that it places a higher weight on lower levels of indices.

We will use integral operator in order to show different orders of stochastic dominance, $\zeta_j(.;G)$, that integrates the function G to order $j-1$.

$\zeta_1(z;G) = G(z)$,

$\zeta_2(z;G) = \int_0^z G(t)dt = \int_0^z \zeta_1(t;G)dt$,

$\zeta_3(z;G) = \int_0^z \int_0^s G(s)dsdt = \int_0^z \zeta_2(t;G)dt$,

and so on.

Following BD, we need the following assumptions:

Assumption 1: Assume that:

(i) F and G have common support $[0, \overline{z}]$ where $\overline{z} \prec \infty$;

(ii) F and G are continuous function on $[0, \overline{z}]$.

These assumptions are required, since the multiple integrals of $CDFs$ will be infinitely large in their absence for $j \geq 2$.

Given the assumptions above, the general hypotheses for testing stochastic dominance of order j can be written compactly as:

$H_0^j : \zeta_j(z;G) \leq \zeta_j(z;F)$ for all $z \in [0, \overline{z}]$,

$H_1^j : \zeta_j(z;G) \succ \zeta_j(z;F)$ for some $z \in [0, \overline{z}]$.

Weak stochastic dominance of any order of G over F implies that G is no larger than at any point of indices (e.g. when we compare 1975 HDI index to 1980 HDI index, if 1980 HDI Index CDF is no larger than 1975 HDI Index). The alternative hypothesis is the converse of the null and implies that there is at least some index value at which G (or its integral) is strictly larger than F (or its integral). In other words stochastic dominance fails at some point for G over F. In order to look F dominance over G, then we can reverse the roles they play in the hypotheses and redoing the tests.

2.2 Test Statistics and Asymptotic Distributions

We have independent samples from the two distributions (e.g. for HDI in 1975 and 1980). In order to allow for different sample sizes we need to make assumptions about the way in which sample sizes grow.

Assumption 2:

(i) $\{X_i\}_{i=1}^{N}$ and $\{Y_i\}_{i=1}^{M}$are independent random samples from distributions with $CDF's$ F and G respectively;

(ii) sampling scheme is such that as $N, M \longrightarrow \infty$, $\frac{N}{N+M} = \phi$ where $0 \prec \phi \prec 1$.

Assumption $2(i)$ deals with the sampling scheme and would be satisfied if one had samples of indices from different segments of a population or separate samples across time. *Assumption* $2(ii)$ implies that the ratio of the sample sizes is finite and bounded away from zero.

The empirical distributions used to construct the tests are respectively,

$\widehat{F}_N(z) = \frac{1}{N}\sum_{i=1}^{N} 1(X_i \leq z), \qquad \widehat{G}_M(z) = \frac{1}{M}\sum_{i=1}^{M} 1(Y_i \leq z).$

The test statistics for testing the hypotheses can be written compactly as follows:

$\widehat{S}_j = \left(\frac{NM}{N+M}\right)^{1/2} \sup_z (\zeta_j(z; \widehat{G}_M) - (\zeta_j(z; \widehat{F}_N)).$

Since ζ_j is a linear operator, then

$$\zeta_j(z; \widehat{F}_N) = \frac{1}{N}\sum_{i=1}^{N} \zeta_j(z; 1_{X_i}) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{(j-1)!} 1(X_i \leq z)(z - X_i)^{j-1} \qquad (2.1)$$

1_{X_i}denotes the function $1(X_i \leq x)$,see Davidson and Duclos (2000).

The limiting distributions of the test statistics under the null hypothesis can be characterized using the fact that

$\sqrt{N}(\widehat{F}_N - F) \Longrightarrow \mathfrak{B}_F \circ F, \qquad \sqrt{M}(\widehat{G}_M - G) \Longrightarrow \mathfrak{B}_G \circ G$

where $\mathfrak{B}_F \circ F$ and $\mathfrak{B}_G \circ G$ are independent Brownian Bridge process.

BD (2003) obtain the characterizing behavior of the test statistics and derive the asymptotic properties of the process that involves integrals of the Brownian Bridges under their Lemma 1 (see the appendix).

We consider tests based on decision rule:

"$reject\ H_0^j\ if\ \ \widehat{S}_j \succ c_j$"

where c_j are suitably chosen critical values to be obtained by simulation methods.

In order to make the result operational, we need to find an appropriate critical value c_j to satisfy $P(\overline{S}_j^F \succ c_j) \equiv \alpha$ or $P(\overline{S}_j^{G,F} \succ c_j) \equiv \alpha$ (some desired probability level such as 0.05 or 0.01). Since the distribution of the test statistic depends on the underlying distribution, this is not an easy task, and we decide hereafter to rely on bootstrap methods to simulate the p-values.

3 Simulating p-values

3.1 Bootstrap Methods

We provide bootstrap methods based on Proposition $A.1(i)$ and $A.1(ii)$. In this case we define sample as $\chi = \{X_1, ..., X_N\}$ and compute the distribution of the random quantity

$$\overline{S}_j^F = \sqrt{N} \sup_z (\zeta_j(z; \widehat{F}_N^*) - \zeta_j(z; \widehat{F}_N)) \quad (3.1)$$

where

$$\widehat{F}_N^*(z) = \frac{1}{N} \sum_{i=1}^{N} (X_i^* \leq z)$$

for a random sample of X_i^* drawn from χ.To simulate the random variable corresponding to $\overline{S}_j^{F,G}$ from Proposition $A.1(ii)$, we use Van der Vaart and Wellner (1996) and resample the combined samples: $\mathfrak{Z} = \{X_1, ..., X_N, Y_1, ..., Y_M\}$. Let $\widehat{G}_M^*$ denote the empirical CDF of a random sample of size M from $\mathfrak{Z}$ and let $\widehat{F}_N^*$ denote the empirical CDF of a random sample of size N from $\mathfrak{Z}$. Then we can compute the distribution of a random quantity

$$\overline{S}_{j,1}^{F,G} = \sqrt{\frac{NM}{N+M}} \sup_z (\zeta_j(z; \widehat{G}_M^*) - \zeta_j(z; \widehat{F}_N)). \tag{3.2}$$

Let $\gamma = \{Y_1, ..., Y_M\}$. Third method of bootstrapping can be done by drawing sample size of N from χ (with replacement) to construct an estimate of $\widehat{F}_N^*$ and drawing samples of size M from γ to construct an estimate of $\widehat{G}_M^*$, so we can compute the statistic

$$\overline{S}_{j,2}^{F,G} = \sqrt{\frac{NM}{N+M}} \sup_z ((\zeta_j(z; \widehat{G}_M^*) - \zeta_j(z; \widehat{G}_M)) - (\zeta_j(z; \widehat{F}_N^*) - \zeta_j(z; \widehat{F}_N))). \tag{3.3}$$

For each case we can do Monte Carlo simulation to simulate the *p-values*. We can denote the *p-values* by the notion $\widetilde{p}_j^F$, $\widetilde{p}_{j,1}^{F,G}$, $\widetilde{p}_{j,2}^{F,G}$ respectively.

Proposition 1: Assuming that $\alpha \prec \frac{1}{2}$, a test for SD_j based on either rule:

"reject H_0^j if $\widetilde{p}_j^F \prec \alpha$,"
"reject H_0^j if $\widetilde{p}_{j,1}^{F,G} \prec \alpha$,"
"reject H_0^j if $\widetilde{p}_{j,2}^{F,G} \prec \alpha$,"
satisfies the following:
$\lim P(reject\ H_0^j) \leq \alpha$ if H_0^j is true,
$\lim P(reject\ H_0^j) = 1$ if H_1^j is true.

The importance of the BD methodology is that it can be applied to different sample sizes over time and even for small sample sizes (e.g. sample size of 50) the power is quite good. In the following section, we provide the empirical results for the HDI index and its components from 1975 to 2000 in 5-year increments.

4 Data and Descriptive Statistics

We use United Nations Development Program's HDI index and its components - life expectancy, education and GDP indices for the period 1975 to

2000 in 5-year increments. Each index ranges between 0 and 1 (from lowest to highest well being). HDI index represents the simple arithmetic average of the three individual indices:

Life expectancy index $= \frac{LE-25}{85-25}$,where the value 85 years gives a life expectancy of 1 and the value of 25 gives a value of 0.

Education index= $\frac{2}{3}$(adult literacy index) + $\frac{1}{3}$(gross enrollment index).

The education index measures country's achievement in both adult literacy rate and combined primary, secondary and tertiary gross enrolment rate.

GDP Index=$\frac{\text{log (GDP per capita) - log (100)}}{\text{log (40,000)-log (100)}}$.

Table 1 presents the descriptive statistics for the overall HDI and the individual component indices over time.

One can see that the HDI index improved over time, as did expectancy and especially education, whereas the GDP per capita index remained almost unchanged same between 1980 and 1995, and the index dropped from 1980 to 1985. We see that the education index increased significantly over this time period without any drop, while the life expectancy index remained steady after 1990. This is mainly because of the drop in life expectancy in Africa. In next the section we will examine the stochastic dominance results for these indices.

5 Results for Stochastic Dominance Tests

We will provide the results of Kolmogorov-Smirnov tests that are based on simulation and bootstrap methods from BD (2003) for stochastic dominance over time for HDI and its components separately. KS1 and KS2 come from (3.2) and (3.3); KSB1, KSB2 and KSB3 come from (3.6), (3.7) and (3.8) respectively. Tables 2i to 2iv present the results for first, second and third stochastic dominance over the period under investigation for HDI and its components: education, life expectancy and GDP index respectively. The vertical column represents the years from 1980 to 2000 that are tested for stochastic dominance against years from 1975 to 1995. Percentage levels in the table represent the significance level of stochastic dominance (e.g. for Table 2i (HDI): 1980 year HDI stochastically dominates the 1975 year in the second and third order sense at the 10 percent level).

Since each component of HDI index uses a fixed weight of one-third to determine the overall HDI, stochastic dominance of HDI over time is driven by stochastic dominance of the individual HDI components (education index, life expectancy index and GDP index). GDP index shows no significant improvement for the whole period between 1975 and 2000. Only the 2000 GDP Index dominates the 1975 GDP Index in both second order and third order sense. The education index in 1985, 1990, 1995 and 2000 dominates the education index in 1975, 1980, 1985 and 1990 in first, second and third order sense respectively (10-year periods). In this case, every country's stock of educational knowledge increased within 10 years. On the other side education index in 1980, 1990 and 2000 dominates the education index in 1975, 1985 and 1995 in second and third order sense (5-year periods). Finally, the life expectancy index in 1985 and 1990 dominates 1975 and 1980 in first, second and third order sense respectively (10 year periods). However life expectancy index in 1995 dominates 1985 in first and second order sense and 2000 dominates 1990 in first order only, whereas 1980 index dominates 1975 index in second and third order sense (5-year period). One can see that first order dominance occurs in life expectancy in a span of 10 years.

The overall HDI in 1985 and 1990 dominates the HDI in 1975 and 1980 in both second and third order sense respectively. HDI Index in 1995 and 2000 dominates the HDI index in 1985 and 1990 in first and second order sense respectively (10 year periods), while the 1980 index dominates the 1975 year index in second and third order sense (5-year period).

It becomes apparent that improvements in HDI index over time are driven by the improvements in education and life expectancy. However, the improvement in education index occurs in second and third order sense in 5-year periods which means that the overall improvement is driven by the education index in a shorter time span.

6 Tests of the optimality of the HDI index

We consider a strictly stationary process $\{\boldsymbol{Y}_t; t \in \mathbb{Z}\}$ taking values in $\mathbb{R}^n$. The observations consist in a realization of $\{\boldsymbol{Y}_t;\ t = 1, ..., T\}$. These data correspond to observed values of n different constituent components of the HDI index ($\boldsymbol{\tau}$). We denote by $F(\boldsymbol{y})$, the continuous cdf of $\boldsymbol{Y} = (Y_1, ..., Y_n)'$ at point $\boldsymbol{y} = (y_1, ..., y_n)'$.

Let us consider a portfolio $\boldsymbol{\lambda} \in \mathbb{L}$ where $\mathbb{L} := \{\boldsymbol{\lambda} \in \mathbb{R}^n_+ : \boldsymbol{e}'\boldsymbol{\lambda} = 1\}$ with $\boldsymbol{e}$ for a vector made of ones. This means that all the different components have positive weight and that the portfolio weights sum to one. Let us denote by $G(z, \boldsymbol{\lambda}; F)$ the cdf of the portfolio value $\boldsymbol{\lambda}'\boldsymbol{Y}$ at point z given by $G(z, \boldsymbol{\lambda}; F) := \int_{\mathbb{R}^n} \mathbb{I}\{\boldsymbol{\lambda}'\boldsymbol{u} \leq z\} dF(\boldsymbol{u})$.

The general hypotheses for testing the optimality of HDI index, hereafter $\boldsymbol{\tau}$, can be written compactly as:

$$
\begin{aligned}
H_0 &: G(z, \boldsymbol{\tau}; F) \leq G(z, \boldsymbol{\lambda}; F) \text{for all } z \in \mathbb{R} \text{ and for all } \boldsymbol{\lambda} \in \mathbb{L}, \\
H_1 &: G(z, \boldsymbol{\tau}; F) > G(z, \boldsymbol{\lambda}; F) \text{for some } z \in \mathbb{R} \text{ or for some } \boldsymbol{\lambda} \in \mathbb{L}.
\end{aligned}
$$

The empirical counterpart to G is simply obtained by integrating with respect to the empirical distribution $\hat{F}$ of F.

We consider the weighted Kolmogorov-Smirnov type test statistic

$$
\hat{S} := \sqrt{T} \frac{1}{T} \sup_{z, \boldsymbol{\lambda}} \left[G(z, \boldsymbol{\tau}; \hat{F}) - G(z, \boldsymbol{\lambda}; \hat{F}) \right],
$$

6.1 Empirical application

The test statistic $\hat{S}$ is derived using mixed integer programming formulations. The following is the full formulation of the model:

$$\max_{\mathbf{z},\boldsymbol{\lambda}} \hat{S} = \sqrt{T}\frac{1}{T}\sum_{t=1}^{T}(L_t - W_t) \tag{6.1a}$$

$$\text{s.t.} M(L_t - 1) \leq z - \boldsymbol{\tau}'\mathbf{Y}_t \leq ML_t, \qquad \forall t \tag{6.1b}$$

$$M(W_t - 1) \leq z - \boldsymbol{\lambda}'\mathbf{Y}_t \leq MW_t, \qquad \forall t \tag{6.1c}$$

$$\boldsymbol{e}'\boldsymbol{\lambda} = 1, \tag{6.1d}$$

$$\boldsymbol{\lambda} \geq 0, \tag{6.1e}$$

$$W_t \in \{0,1\},\ L_t \in \{0,1\}, \qquad \forall t \tag{6.1f}$$

with M being a large constant.

The model is a mixed integer program maximizing the distance between the sum over all scenarios of two binary variables, $\frac{1}{T}\sum_{t=1}^{T} L_t$ and $\frac{1}{T}\sum_{t=1}^{T} W_t$ which represent $G(z,\boldsymbol{\tau};\hat{F})$ and $G(z,\boldsymbol{\lambda};\hat{F})$, respectively (the empirical cdf of $\boldsymbol{\tau}$ and $\boldsymbol{\lambda}$ at point z). According to inequalities (6.1b), L_t equals 1 for each scenario $t \in T$ for which $z \geq \boldsymbol{\tau}'\mathbf{Y}_t$, and 0 otherwise. Analogously, inequalities (6.1c) ensure that W_t equals 1 for each scenario for which $z \geq \boldsymbol{\lambda}'\mathbf{Y}_t$. Equation (6.1d) defines the sum of all portfolio weights to be unity, while inequality (6.1e) disallows for short positions in the available assets.

This formulation permits to test the dominance of the HDI index ($\boldsymbol{\tau}$) over any potential linear combination $\boldsymbol{\lambda}$ of the components.

When some of the variables are binary, corresponding to mixed integer programming, the problem becomes NP-complete (non-polynomial, i.e., formally intractable).

We reformulate the problem in order to reduce the solving time and to obtain a tractable formulation. The steps are the following:

1) The factor $\sqrt{T}/T$ can be left out in the objective function, since T is

fixed.

2) We can see that there is a set of at most T values, say $\mathcal{R} = \{r_1, r_2, ..., r_T\}$, containing the optimal value of the variable z.

Proof: Vectors $\boldsymbol{\tau}$ and $\mathbf{Y}_t$, $t = 1, ..., T$ being given, we can rank the values of $\boldsymbol{\tau}'\mathbf{Y}_t, t = 1, ..., T$, by increasing order. Let us call $r_1, ..., r_T$ the possible different values of $\boldsymbol{\tau}'\mathbf{Y}_t$, with $r_1 < r_2 < ... < r_T$ (actually there may be less than T different values). Now, for any z such that $r_i \leq z \leq r_i + 1$, $\sum_{t=1,...,T} L_t$ is constant (it is equal to the number of t such that $\boldsymbol{\tau}'\mathbf{Y}_t \leq r_i$). Further, when $r_i \leq z \leq r_i + 1$, the maximum value of $-\sum_{t=1,...,T} W_t$ is reached for $z = r_i$. Hence, we can restrict z to belong to the set $\mathcal{R}$.

3) A direct consequence is that we can solve the original problem by solving the smaller problems $P(r)$, $r \in \mathcal{R}$, in which z is fixed to r. Then we take take the value for z that yields the best total result. The advantage is that the optimal values of the L_t variables are known in $P(r)$. Precisely, $\sum_{t=1,...,T} L_t$ is equal to the number of t such that $\boldsymbol{\tau}'\mathbf{Y}_t \leq r$. Hence problem $P(r)$ boils down to:

$$\begin{aligned} &\min \sum_{t=1}^{T} W_t \\ &\text{s.t.} M(W_t - 1) \leq r - \boldsymbol{\lambda}'\mathbf{Y}_t \leq MW_t, \quad \forall t \in T \\ &\quad \boldsymbol{e}'\boldsymbol{\lambda} = 1, \\ &\quad \boldsymbol{\lambda} \geq 0, \\ &\quad W_t \in \{0, 1\}, \quad \forall t \in T. \end{aligned} \tag{6.2a}$$

Note that this becomes a minimization problem.

Problem $P(r)$ amounts to find the largest set of constraints $\boldsymbol{\lambda}'\mathbf{Y}_t \geq r$ consistent with $\boldsymbol{e}'\boldsymbol{\lambda} = 1$ and $\boldsymbol{\lambda} \geq 0$.

Let $M_t = \min \mathbf{Y}_{t,i}, i = 1, ..., n$, i.e., the smallest entry of vector $\mathbf{Y}_t$.

Clearly, for all $\boldsymbol{\lambda} \geq 0$ such that $\boldsymbol{e}'\boldsymbol{\lambda} = 1$, we have that $\boldsymbol{\lambda}'\mathbf{Y}_t \geq M_t$. Hence, Problem $P(r)$ can be rewritten in an even better reduced form:

$$\begin{aligned}
&\min \sum_{t=1}^{T} W_t \\
&\text{s.t.} \boldsymbol{\lambda}'\mathbf{Y}_t \geq r - (r - M_t)W_t, \quad \forall t \in T \\
&\quad \boldsymbol{e}'\boldsymbol{\lambda} = 1, \\
&\quad \boldsymbol{\lambda} \geq 0, \\
&\quad W_t \in \{0, 1\}, \quad \forall t \in T.
\end{aligned} \tag{6.3a}$$

We further simplify $P(r)$ by fixing the following variables:

- for all t such that $r \leq M_t$, the optimal value of W_t is equal to 0 since the half space defined by the t-th inequality contains the simplex.

- for all t such that $r \geq M_t$, the optimal value of W_t is equal to 1 since the half space defined by the t-th inequality has an empty intersection with the simplex.

6.2 Results

The computational time for this mixed integer programming formulation is large. For the optimal solution (which involves 1264 mixed integer optimization programs, one for each discrete value of z) it takes less than two days. The problems are optimized with IBM's OSL solver on an IBM workstation (with a 2*2.4 GHz Power, 6Gb of RAM). We note the almost exponential increase in solution time with the increasing number of observations. We stress here the computational burden that is managed for these tests. The optimization problems are modelled in GAMS. The General Algebraic Modeling System (GAMS) is a high-level modeling system for mathematical programming and optimization. It consists of a language compiler and a stable of integrated high-performance solvers. GAMS is tailored for complex, large scale modeling applications. OSL uses the branch and bound technique to solve the MIP program.

We found that the HDI index is not optimal. We can construct many other portfolios $\boldsymbol{\lambda}$ consisting of the three components of the HDI index (life expectancy, educational attainment, and standard of living) that dominate the HDI index, for which $G(z, \boldsymbol{\tau}; F) > G(z, \boldsymbol{\lambda}; F)$. The greatest difference is for portfolio $\boldsymbol{\lambda} = (0, 1, 0)$ at point z=0.90107.

Table 3 summarizes the results. In the first row we see the greatest distance (0.04540) between the optimal portfolio $\boldsymbol{\lambda}$ and the HDI index, which is achieved for $z = 0.90717$. Next, the Table presents the 20 portfolios closest to the optimal. It is clear that the educational knowledge has the greatest impact, since in all portfolios participates with more that 97%.

The above result suggests that using the fixed equal weight results in a suboptimal index that is dominated by many other potential hybrids with different weights. However, the most interesting result we obtain is that using the education index *alone* would dominate any other index that uses all three components. Educational knowledge has the greatest impact in the construction of an optimal HDI index and it greatly dominates all the other two components (with at least a 0.97 weight in the top 21 optimal hybrid indices as seen in Table 3) That is consistent with Sen (1987), who makes the distinction between welfare measures that are inputs such as education and outputs such as health and/or income. Welfare comparisons based on the former measures are preferable to the ones based on the latter.

7 Conclusion

In this paper we consider consistent tests for stochastic dominance efficiency at dominance order. We consider consistent tests, that are similar to Kolmogorov-Smirnov tests and use a variety of approaches to inference based on simulation and the bootstrap. The empirical application to HDI index and its components show that education and life expectancy indices are the main determining factor for the improvement in HDI index over years. We show that the significant indices that has improved within 10-year periods are education index and life expectancy. Moreover, education index has second and third order stochastic dominance within 5 year periods, which means that not all countries have better education index, but there is an overall improvement in educational knowledge in 5 year periods. GDP Index has no significant improvement over 25 years.

Moreover and more importantly, when we consider consistent tests for stochastic dominance efficiency at any order of a given hybrid index with respect to all possible indices constructed from a set of individual components we find that the fixed equal weighted HDI is not optimal and it is educational knowledge alone that has the greatest impact in the construction of an optimal HDI.

8 References

Anderson, G. (1996) “Nonparametric Tests of Stochastic Dominance in Income Distributions,” *Econometrica*, 64, 1183-1193.

Barrett, G., and S. Donald (2003) “Consistent Tests for Stochastic Dominance,” *Econometrica*, 71, 71-104.

Beach, C., and R. Davidson (1983) “Distribution-free Statistical Inference with Lorenz Curves and Income Shares,” *Review of Economic Studies*, 50, 723-735.

Becker, G. S., and T.J. Philipson, and R.R. Soares (2005) "The Quantity and Quality of Life and the Evolution of World Inequality," *American Economic Review*, 95, 277-91.

Bourguignon, F., and C. Morrisson (2002) "Inequality among World Citizens: 1820-1992," *American Economic Review*, 92, 727-744.

Davidson, R., and J.-Y. Duclos (2000) “Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality,” *Econometrica*, 68, 1435-1464.

Easterlin (1995) "Will Raising the Incomes of All Increase the Happiness of All?," *Journal of Economic Behavior and Organization*, 27, 35-48.

Hansen, B. E. (1996) “Inference When a Nuisance Parameter Is Not Identified under the Null Hypothesis,” *Econometrica*, 64, 413–430.

Lehmann, E. L. (1986) *Testing Statistical Hypotheses*, 2nd Edition. New York: John Wiley and Sons.

Maasoumi E. (1999) "Multidimensional Approaches to Welfare Analysis," Chapter 15, in J. Silber (ed.) Handbook of Income Inequality Measurement, Kluwer publishers, 1999, (With Comments by F. Bourguignon), 437-484.

Quah, D. (1996) "Twin Peaks: Growth and Convergence in Models of Distribution Dynamics" *Economic Journal,* 106, 1045-1055.

Ponthiere G. (2004): "Incorporating Changes in Life Expectancy into Economic Growth Rates: an Application to Belgium, 1867-1997," CREPP Working Papers.

Pradhan, M., Sahn D.E. and S. D. Younger (2003) "Decomposing World Health Inequality" *Journal of Health Economics*, 22, 271-293.

Ravallion M. (1994): *Poverty Comparisons, Fundamentals of Pure and Applied Economics. Switzerland.* Harwood Academic Publishers.

Sala-I-Martin X. (2006): "The World Distribution of Income: Falling Poverty and...Convergence Period," *Quarterly Journal of Economics*, 121, 351-397.

Scaillet O., and N. Topaloglou (2006): "Testing for Stochastic Dominance Efficiency," *Journal of Business and Economic Statistics*, forthcoming.

Sen, A. (1987) *The Standard of Living: The Tanner Lectures*, Cambridge University Press: Cambridge UK.

Van Der Vaart, A. W., and J. A. Wellner (1996): *Weak Convergence and Empirical Processes: With Applications to Statistics.* New York: Springer-Verlag.

9 Appendix

Lemma 1: BD show that for $j \geq 2$,

$\sqrt{N}(\zeta_j(.;\widehat{F}_N) - \zeta_j(.;F)) \Longrightarrow \zeta_j(.;\mathfrak{B}_F \circ F)$ in $C([0,\overline{z}])$ where the limit process is with mean zero and covariance kernel given by $(for\ z_2 \succ z_1)$

$$\Omega_j(z_1, z_2, F) = E(\zeta_j(z_1;\mathfrak{B}_F \circ F)\zeta_j(z_2;\mathfrak{B}_F \circ F))$$
$$= \sum_{l=0}^{j-1} \theta_l^j \tfrac{1}{l!}(z_2 - z_1)^l \zeta_{2j-l-1}(z_1;F) - \zeta_j(z_1;F)\zeta_j(z_2;F)$$

where

$\theta_l^j = \binom{2j-l-2}{j-1}$.

Note that this process also holds for G. Lemma 1 provides the covariance kernel in terms of the coefficients θ_l^j and the integration operators that is useful in what follows.

We consider tests based on decision rule:

$"reject\ H_0^j\ if\ \ \widehat{S}_j \succ c_j"$

where c_j is some critical value that will be discussed in a moment.

The following result characterizes the properties of tests, where

$$\overline{S}_j^F = \sup_z \zeta_j(z;\mathfrak{B}_F \circ F)$$
$$\overline{S}_j^{G,F} = \sup_z(\sqrt{\phi}\zeta_j(z;\mathfrak{B}_G \circ G) - \sqrt{1-\phi}\zeta_j(z;\mathfrak{B}_F \circ F))$$

Proposition A.1: Let c_j be a positive finite constant, then:

$A(i)\ if\ H_0^j\ is\ true,$

$$\lim_{N,M\to\infty} P(reject\ H_0^j) \leq P(\overline{S}_j^F \succ c_j) \equiv \alpha_F(c_j)$$

with equality when $F(z) = G(z)$ for all $z \in [0,\overline{z}]$;

$A(ii)\ if\ H_0^j\ is\ true,$

$$\lim_{N,M\to\infty} P(reject\ H_0^j) \leq P(\overline{S}_j^{G,F} \succ c_j) \equiv \alpha_{G,F}(c_j)$$

with equality when $F(z) = G(z)$ for all $z \in [0,\overline{z}]$;

$B\ if\ H_0^j\ is\ false,$

$$\lim_{N,M\to\infty} P(reject\ H_0^j) = 1.$$

The result provides a random variable that dominates the limiting random variable corresponding to the test statistic under the null hypothesis.

The inequality yield a test that will never reject more often than $\alpha_F(c_j)$ (respectively $\alpha_{G,F}(c_j)$) for any G satisfying the null hypothesis. As noted in the result the probability of rejection will asymptotically be exactly $\alpha_F(c_j)$ when $F = G$ ($\alpha_{G,F}(c_j)$ respectively), and, moreover, $\alpha_F(c_j) = \alpha_{G,F}(c_j)$ because of the fact that $\overline{S}_j^{G,F} \overset{d}{=} \overline{S}_j^F$.The results in A part implies that if one could find a c_j to set the $\alpha_F(c_j)$ (respectively $\alpha_{G,F}(c_j)$) to some desired probability level (say the conventional 0.05 or 0.01) then this would be the significance level for composite null hypotheses in the sense described by Lehmann (1986). The result in B part indicates that the test is capable of detecting any violation of the full set of restrictions of the null hypothesis. Of course, in order to make the result operational, we need to find an appropriate critical value c_j to satisfy $P(\overline{S}_j^F \succ c_j) \equiv \alpha$ or $P(\overline{S}_j^{G,F} \succ c_j) \equiv \alpha$. Since the distribution of the test statistic depends on the underlying distribution, this is not an easy task, and we rely on numerical simulation methods to simulate p-values such as the bootstrap.

Table 1: Data and Descriptive Statistics

Table 1i: HDI index						
	1975	1980	1985	1990	1995	2000
Sample	101	113	121	136	146	138
Mean	0.5975	0.6345	0.6480	0.6719	0.6826	0.7006
Median	0.6153	0.6606	0.6835	0.7082	0.7270	0.7409
Std. Dev.	0.1969	0.1886	0.1856	0.1831	0.1834	0.1818

Table1ii: Education Index						
	1975	1980	1985	1990	1995	2000
Sample	101	113	121	136	146	138
Mean	0.5936	0.6393	0.6693	0.7089	0.7348	0.7643
Median	0.6170	0.6845	0.7334	0.7799	0.8071	0.8317
Std. Dev.	0.2602	0.2392	0.2279	0.2171	0.2074	0.1958

Table 1iii: Life Expectancy Index						
	1975	1980	1985	1990	1995	2000
Sample	101	113	121	136	146	138
Mean	0.5745	0.6171	0.6368	0.6600	0.6650	0.6734
Median	0.5732	0.6300	0.6635	0.7121	0.7177	0.7411
Std. Dev.	0.1818	0.1736	0.1739	0.1752	0.1880	0.2011

Table 1iv: GDP Index						
	1975	1980	1985	1990	1995	2000
Sample	101	113	121	136	146	138
Mean	0.6245	0.6472	0.6378	0.6469	0.6479	0.6642
Median	0.6100	0.6421	0.6309	0.6435	0.6486	0.6623
Std. Dev.	0.1848	0.1832	0.1866	0.1898	0.1932	0.1960

Table 2: Stochastic Dominance Results

Table 2i: Stochastic Dominance Results for HDI index						
		1975	1980	1985	1990	1995
1980	SD1	N/A	-	-	-	-
	SD2	10%	-	-	-	-
	SD3	10%	-	-	-	-
1985	SD1	N/A	N/A	-	-	-
	SD2	1-5%	N/A	-	-	-
	SD3	1-5%	N/A	-	-	-
1990	SD1	1%	N/A	N/A	-	-
	SD2	1-5%	5-10%	N/A	-	-
	SD3	1-5%	5-10%	N/A	-	-
1995	SD1	1%	10%	10%	N/A	-
	SD2	1%	5%	10%	N/A	-
	SD3	1%	1-5%	10%	N/A	-
2000	SD1	1%	5%	1%	10%	N/A
	SD2	1%	1%	1%	10%	N/A
	SD3	1%	1%	1-5%	N/A	N/A

Table 2ii: Stochastic Dominance Results for Education Index						
		1975	1980	1985	1990	1995
1980	SD1	N/A	-	-	-	-
	SD2	10%	-	-	-	-
	SD3	5-10%	-	-	-	-
1985	SD1	5%	N/A	-	-	-
	SD2	1-5%	N/A	-	-	-
	SD3	1%	N/A	-	-	-
1990	SD1	1%	10%	N/A	-	-
	SD2	1%	1%	5-10%	-	-
	SD3	1%	1%	5-10%	-	-
1995	SD1	1%	5%	5%	N/A	-
	SD2	1%	1%	1%	N/A	-
	SD3	1%	1%	1%	N/A	-
2000	SD1	1%	1%	1%	5%	N/A
	SD2	1%	1%	1%	1-5%	10%
	SD3	1%	1%	1%	1-5%	10%

Table 2iii: Stochastic Dominance Results for Life Expectancy Index						
		1975	1980	1985	1990	1995
1980	SD1	N/A	-	-	-	-
	SD2	1-5%	-	-	-	-
	SD3	5%	-	-	-	-
1985	SD1	5%	N/A	-	-	-
	SD2	1%	N/A	-	-	-
	SD3	1%	N/A	-	-	-
1990	SD1	1%	5%	N/A	-	-
	SD2	1%	1-5%	N/A	-	-
	SD3	1%	5-10%	N/A	-	-
1995	SD1	1%	1%	5%	N/A	-
	SD2	1%	1-5%	10%	N/A	-
	SD3	1%	10%	N/A	N/A	-
2000	SD1	1%	1%	5%	10%	N/A
	SD2	1%	1%	10%	N/A	N/A
	SD3	1%	10%	N/A	N/A	N/A

Table 2iv: Stochastic Dominance Results for GDP index						
		1975	1980	1985	1990	1995
1980	SD1	N/A	-	-	-	-
	SD2	N/A	-	-	-	-
	SD3	N/A	-	-	-	-
1985	SD1	N/A	N/A	-	-	-
	SD2	N/A	N/A	-	-	-
	SD3	N/A	N/A	-	-	-
1990	SD1	N/A	N/A	N/A	-	-
	SD2	N/A	N/A	N/A	-	-
	SD3	N/A	N/A	N/A	-	-
1995	SD1	N/A	N/A	N/A	N/A	-
	SD2	N/A	N/A	N/A	N/A	-
	SD3	N/A	N/A	N/A	N/A	-
2000	SD1	N/A	N/A	N/A	N/A	N/A
	SD2	5-10%	N/A	N/A	N/A	N/A
	SD3	10%	N/A	N/A	N/A	N/A

Table 3: Optimal Portfolios for HDI index				
		Optimal Portfolio		
Distance	z	lexindex	eduindex	gdpindex
0.04540	0.90717	0.00000	1.00000	0.00000
0.04539	0.74382	0.00021	0.99979	0.00000
0.04538	0.69882	0.00000	0.99965	0.00035
0.04537	0.71781	0.00000	0.99957	0.00043
0.04535	0.78122	0.00078	0.99922	0.00000
0.04531	0.73016	0.00129	0.99871	0.00000
0.04528	0.74890	0.00000	0.99825	0.00175
0.04525	0.78005	0.00220	0.99780	0.00000
0.04519	0.73529	0.00312	0.99688	0.00000
0.04515	0.84347	0.00000	0.99633	0.00367
0.04509	0.71036	0.00000	0.99550	0.00450
0.04500	0.70385	0.00586	0.99414	0.00000
0.04490	0.71098	0.00000	0.99266	0.00734
0.04480	0.68633	0.00000	0.99112	0.00888
0.04470	0.74293	0.01021	0.98979	0.00000
0.04460	0.72715	0.01171	0.98829	0.00000
0.04450	0.70816	0.00242	0.98682	0.01076
0.04439	0.62463	0.01484	0.98516	0.00000
0.04430	0.70862	0.00000	0.98379	0.01621
0.04400	0.60209	0.01594	0.97939	0.00467
0.04370	0.70088	0.01554	0.97502	0.00944
Optimal portfolios of life expectancy (lexindex), educational attainment (eduindex) and standard of living (gdpindex) that dominate the fixed equal weighted HDI index.				